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V. I. Arnol'd (Ed.)

Dynamical Systems V

Bifurcation Theory and Catastrophe Theory

动力系统 V

分歧理论和突变理论

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I. Bifurcation Theory

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Yu. S. Il'yashenko, L.P. Shil'nikov

Translated from the Russian
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Preface

The word "bifurcation" means "splitting into two". "Bifurcation" is used to describe any sudden change that occurs while parameters are being smoothly varied in any system: dynamical, ecological, etc. Our survey is devoted to the bifurcations of phase portraits of differential equations – not only to bifurcations of equilibria and limit cycles, but also to perestroikas of the phase portraits of systems in the large and, above all, of their invariant sets and attractors. The statement of the problem in this form goes back to A.A. Andronov.

Connections with the theory of bifurcations penetrate all natural phenomena. The differential equations describing real physical systems always contain parameters whose exact values are, as a rule, unknown. If an equation modeling a physical system is structurally unstable, that is, if the behavior of its solutions may change qualitatively through arbitrarily small changes in its right-hand side, then it is necessary to understand which bifurcations of its phase portrait may occur through changes of the parameters.

Often model systems seem to be so complex that they do not admit meaningful investigation, above all because of the abundance of the variables which occur. In the study of such systems, some of the variables that change slowly in the course of the process described are, as a rule, assumed to be constant. The resulting system with a smaller number of variables can then be investigated. However, it is frequently impossible to consider the individual influences of the discarded terms in the original model. In this case, the discarded terms may be looked upon as typical perturbations, and, accordingly, the original model can be described by means of bifurcation theory applied to the reduced system.

Reformulating the well-known words of Poincaré on periodic solutions, one may say that bifurcations, like torches, light the way from well-understood dynamical systems to unstudied ones. L.D. Landau, and later E. Hopf, using this idea of bifurcation theory, offered a heuristic description of the transition from laminar to turbulent flow as the Reynolds number increases. In Landau's scenario this transition was accomplished through bifurcations of tori of steadily growing dimensions. Later on when the zoo of dynamical systems and their bifurcations had significantly grown, many papers appeared, describing – mainly at a physical level – the transition from regular (laminar) flow to chaotic (turbulent) flow. The chaotic behavior of the 3-dimensional model of Lorenz for convective motions has been explained with the aid of a chain of bifurcations. This explanation is not included in the present survey since, to save space, bifurcations of systems with symmetry have not been included. Lorenz's system is centrally symmetric.

The theory of relaxation oscillations, which deals with systems in which the parameters slowly change with time (these parameters are called slow variables), closely adjoins the theory of bifurcations in which parameters do not change with time. In "fast-slow" systems of relaxation oscillations, a slowness parameter enters that characterizes the speed of change of the slow variables. When this parameter is zero, a fast-slow system transforms into a family studied in the theory of bifurcations, but at a nonzero value of the parameter specific phenomena arise which are sometimes called dynamical bifurcations.

In this survey, systematic use is made of the theory of singularities. The solutions to many problems of bifurcation theory (mostly of local ones) consist of presenting and investigating a so-called principal family – a kind of topological normal form for families of the class studied. The theory of singularities helps to guess at, and partially to investigate, principal families. This theory also describes the theory of bifurcations of equilibrium states, singularities of slow surfaces, slow motions in the theory of relaxation oscillations, etc.

We also note that finitely smooth normal forms of local families of differential equations are especially useful in the theory of nonlocal bifurcations. On one hand, these normal forms substantially simplify the presentation and investigation of bifurcations, and also simplify and clarify the proof and analysis of the results obtained. On the other hand, the nonlocal theory of bifurcations helps to select problems from the theory of normal forms that are important for applications. In our opinion, at the present time, the connection between the theory of normal forms and the nonlocal theory of bifurcations is not used often enough.

This survey includes, along with what is known, a series of new results, some of these are known to the authors through private communications. [Added in translation: The results mentioned below were new when the Russian text was written (1985). Now most of them have been published. The additional list of references is given after the main one and numbered.] Among these are eight new topics. The first is a complete investigation of bifurcations from equilibria in generic two-parameter families of vector fields on the plane with two intersecting invariant curves (the so-called reduced problem for two purely imaginary pairs, Sect. 4.5 and Sect. 4.6 of Chap. 1 (see Żołądek (1987)). The second is the construction of finitely smooth normal forms and functional moduli of the C^1-classification of local families of vector fields and diffeomorphisms (Yu.S. Il'yashenko and S.Yu. Yakovenko, Sect. 5.7–5.10 of Chap. 2 (see Il'yashenko and Yakovenko [3*, 4*])). The third is the construction of a topological invariant of vector fields with a trajectory homoclinic to a saddle with complex eigenvalues (Sect. 5.6 of Chap. 3). The fourth is the description of a generic two-parameter deformation of a vector field with two homoclinic curves at a saddle, in which the bifurcation diagram of the deformation contains a continuum of components. (D.V. Turaev and L.P. Shil'nikov [9*], Sect. 7.2 of Chap. 3). The fifth result is the definition of a statistical limit set as a possible candidate for the concept of a physical attractor (Sect. 8.2 of Chap. 3 (Il'yashenko [2*])). The sixth one is the description of connections between the theory of implicit equations and relaxation oscillations, and the normalization of slow motions for fast-slow systems with one or two slow variables (see Arnol'd's theorem in Sect. 2.2–2.7 of Chap. 4 and the related paper by Davidov [1*]). The seventh result is normalization of fast-slow equations, and the explicit form and investigation of systems of first approximation (Sect. 3.2–3.5 of Chap. 4; see the related paper by Teperin [8*]). The eighth and last one is the investigation of the delayed loss of stability in generic fast-slow systems as a pair of eigenvalues of a stable singular point of a fast equation crosses the imaginary axis (the birth of a cycle as a dynamical bifurcation (A.I. Nejshtadt, § 4 of Chap. 4); see [6*, 7*]). We also point here to a conjecture on the bifurcations in generic multiple parameter families of vector fields on the plane that is closely related to Hilbert's 16^{th} problem (Sect. 2.8 of Chap. 3).

Our survey, inevitably, is incomplete. We did not include in it the comparatively few works on local bifurcations in three-parameter families and on nonlocal bifurcations in two-parameter families; some relevant citations are, however, given in the References. In describing nonlocal bifurcations we limited ourselves to only those things which happen on the boundary of the set of Morse-Smale systems. The theory of such bifurcations is substantially complete, although it is not very well known; it is mostly due to works of the Gor'kij school, which often have been published in sources that are hard to obtain. That part of the boundary of the set of Morse-Smale systems on which a countable set of nonwandering trajectories arise is not yet fully explored; but Sect. 7 of Chap. 3 is devoted to this problem. For reasons of consistency of style we often formulate known results in a form different from that in which they first appeared.

Chap. 1 and 2 were written by V.I. Arnol'd and Yu.S. Il'yashenko. Chap. 3, in its final version, was written by V.S. Afrajmovich and Yu.S. Il'yashenko with the participation of V.I. Arnol'd and L.P. Shil'nikov. Sect. 1.6 of Chap. 2 was written by V.S. Afrajmovich. Sects. 1 and 2 of Chap. 4 were written by V.I. Arnol'd, Sect. 3, except for Sect. 3.7, by Yu.S. Il'yashenko. Sect. 3.7 was written by N.Kh. Rozov, Sect. 4 by A.I. Nejshtadt, Sect. 5 by A.K. Zvonkin; the authors sincerely thank them. The authors do not claim that the list of References is complete. In its organization we followed the same principles as in the survey by Arnol'd and Il'yashenko (1985). The symbol ▲ denotes the end of some formulations.

Chapter 1
Bifurcations of Equilibria

The theory of bifurcations of dynamical systems describes sudden qualitative changes in the phase portraits of differential equations that occur when parameters are changed continuously and smoothly. Thus, upon loss of stability, a limit cycle may arise from a singular point, and the loss of stability by a limit cycle may give rise to chaos. Such changes are termed bifurcations.

In Chap. 1 and 2 only local bifurcations are investigated, that is, bifurcations of phase portraits near singular points and limit cycles are considered.

In differential equations describing real physical phenomena, singular points and limit cycles are most often found in general position, that is, they are hyperbolic. However, there are special classes of differential equations where matters stand differently. Such classes are, for example, systems having symmetries related to the very nature of the phenomena investigated, and also Hamiltonian systems, reversible systems, and equations that preserve phase volume. Consider, for example, the one-parameter family of dynamical systems on the line with second-order symmetry:

$$\dot{x} = v(x, \varepsilon), \qquad v(-x, \varepsilon) = -v(x, \varepsilon).$$

A typical bifurcation of a symmetric equilibrium in such a system is the *pitchfork bifurcation* shown in Fig. 1 ($v = x(\varepsilon - x^2)$). In this bifurcation, from the loss of stability by a symmetric equilibrium, two new, less symmetric, equilibria branch out. In this process the symmetric equilibrium position continues to exist, but it loses its stability.

In typical one-parameter families of general (nonsymmetric) systems, pitchfork bifurcations do not occur. Under a small perturbation of the vector field $v(x, \varepsilon)$ above (although the breaking of symmetry may be ever so slight) the pitchfork in Fig. 1 changes into one of the four pairs of curves in Fig. 2. From these pictures it is evident that the phenomena occurring in response to a smooth, slow change of a parameter in an idealized, strictly symmetric system are qualitatively different from those in a perturbation of it. Therefore, it is necessary to take account of the influence of a slight breaking of symmetry when analysing bifurcations in symmetric systems, if such a break is generally possible. On the other hand, strictly symmetric models occur in some instances. Such is the case, for example, for normal forms (see §3 below). In these cases it is necessary to investigate bifurcations of symmetric systems within the class of perturbations that do not break symmetry.

The degenerate cases which are avoidable by small generic perturbations of an individual system may become unavoidable when families of systems are studied. Therefore, in the investigation of degenerate cases, instead of studying an individual degenerate equation one should always consider the bifurcations that occur in generic families of systems that display a similar degeneracy in an

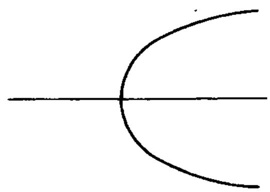

Fig. 1. Bifurcation of equilibria in a symmetric system

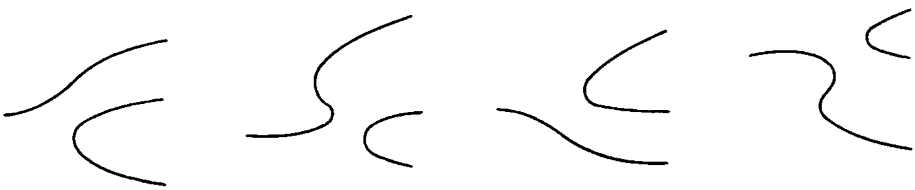

Fig. 2. Bifurcation of equilibria in a nearly symmetric system

unavoidable form. Technically, this investigation is carried out with the help of the construction of special, so-called versal, deformations; in some sense these contain all possible deformations.

§ 1. Families and Deformations

In this section the transversality theorem and the "reduction principle", which allows one to lower the dimension of phase space by "neglecting" inessential (hyperbolic) variables, are formulated.

1.1. Families of Vector Fields. We consider a family of differential equations, say,

$$\dot{x} = v(x, \varepsilon), \qquad x \in U \subset \mathbb{R}^n, \qquad \varepsilon \in B \subset \mathbb{R}^k.$$

The domain U is called *phase space*, B is called the *space of parameters* (or the *base* of the family), and v is called a family of vector fields on U with base B. Henceforth, unless stated otherwise, only smooth families will be considered (v is of class C^∞).

1.2. The Space of Jets. Let U and W be domains of the real, linear spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If we choose coordinate systems in $\mathbb{R}^n$ and $\mathbb{R}^m$, then the k-jet of a mapping $U \to W$ at a point x is the vector-valued Taylor polynomial at x with degree $\leqslant k$. Similarly, the set of all k-jets of mappings $U \to W$ is defined by $U \times \{$the space of m-component vector polynomials, of degrees no greater than k, in n variables, with constant terms in $W\}$, and therefore it is a smooth manifold. The *manifold of k-jets of mappings* $U \to W$ is denoted by $J^k(U, W)$.

Analogously, $J^k(M, N)$ is the manifold of k-jets of mappings of a smooth manifold M into a smooth manifold N.

1.3. Sard's Lemma and Transversality Theorems. Consider a smooth mapping $f: U \to W$. A point x of U is *regular* if the image, under the derivative of f at x, of the tangent space at x is the whole tangent space to W:

$$f_*(x)T_xU = T_{f(x)}W.$$

The value of f at a *critical* (i.e., nonregular) point is called a *critical value.*

Sard's Lemma. *The set of critical values of a smooth mapping has Lebesgue measure zero.*

Definition. Two linear subspaces X and Y of a linear space L are *transversal* if their sum is the whole space: $X + Y = L$. [For example, two perpendicular planes in $\mathbb{R}^3$ are transversal, two perpendicular straight lines are not. Translator]

Everywhere in this subsection A and B denote smooth manifolds, and C is a smooth submanifold of B.

Definition. The mapping $f: A \to B$ is called *transversal to C at a point a* in A if either $f(a)$ does not belong to C or the tangent plane to C at $f(a)$ and the image, under the derivative of f at a, of the tangent plane to A at a are transversal:

$$f_*(a)T_aA + T_{f(a)}C = T_{f(a)}B.$$

Definition. The mapping $f: A \to B$ is *transversal* to C if it is transversal to C at each point of A.

Remark. If $\dim A + \dim C < \dim B$ and a mapping $f: A \to B$ is transversal to C, then the intersection $f(A) \cap C$ is empty.

We denote by $C^r(U, W)$ the space of r-smooth mappings of U into W.

The Weak Transversality Theorem for Domains in $\mathbb{R}^n$. *Let C be a smooth submanifold in W. The mappings $f: U \to W$ that are transversal to C form an everywhere dense countable intersection of open sets*[1] *in $C^r(U, W)$ (where $r >$* max $(\dim W - \dim U - \dim C, 0)$).

The Weak Transversality Theorem for Manifolds. *Let A be a compact manifold, and let C be a compact submanifold of a manifold B. Then the mappings $f: A \to B$ transverse to C form an open everywhere dense set in the space of all r-smooth mappings of A into B (where $r >$* max $(\dim B - \dim A - \dim C, 0)$).

Remarks. The closeness of two mappings is defined in terms of the C^r-norms of the functions determining them. If one of the manifolds A or C is not compact, then "open everywhere dense set" must be replaced by "residual set".

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[1] Such intersections are sometimes called *thick sets* or *residual sets.*

Let M and N be smooth manifolds (or domains in vector spaces). Associated to each smooth mapping is its '*k-jet extension*' $j^k f\colon M \to J^k(M, N)$; the *k-jet* of the mapping f at x corresponds to a point x of M.

Thom's Transversality Theorem. *Let C be a proper submanifold of the space of k-jets $J^k(M, N)$. Then the set of mappings $f\colon M \to N$, whose k-jet extensions are transversal to C, forms a residual set in the space of mappings from M into N in the* C^r-topology (*where* $r \geqslant r_0(k, \dim M, \dim N)$, *for some function* r_0).

1.4. Simplest Applications: Singular Points of Generic Vector Fields. Everywhere in this subsection a "generic" field or family is a field or family from some residual subset of the corresponding function space. Vector fields are defined on domains of the space $\mathbb{R}^n$.

Theorem. *For a generic family of vector fields the set of singular points of the fields of the family forms a smooth submanifold in the direct product of phase space with the space of parameters.*

◂ The set of singular points of the fields of family has the form $\{(x, \varepsilon) | v(x, \varepsilon) = 0\}$. By Sard's lemma the set of critical values of the mapping v has measure zero. Consequently, there exists an arbitrarily small vector δ, for which $-\delta$ is a regular value of the mapping v. The set $\{v(x, \varepsilon) = -\delta\}$ is a smooth submanifold by the implicit function theorem. But this submanifold is the set of singular points of vector fields of the family $v(x, \varepsilon) + \delta$. ▸

The projection of the manifold of equilibria onto the space of parameters is a smooth mapping. The theory of singularities of smooth mappings (in particular, of projections) allows one to classify the critical points of generic mappings (and, consequently, also the bifurcations of equilibrium positions in generic families).

For example, if there is just one parameter, then a typical bifurcation is, modulo diffeomorphisms fibred over the axis of parameters, the same as in the family with equilibrium curve $\varepsilon = \pm x^2$ (birth or death of a pair of equilibria). If there are two parameters, then projection leads to one of the normal forms:

$$\varepsilon_1 = \pm x^2 \qquad \text{(a fold)},$$

$$\varepsilon_1 = x^3 \pm \varepsilon_2 x \quad \text{(a Whitney pleat or cusp)}.$$

Theorem. *All the singular points of a generic vector field are nondegenerate* (*do not have zero eigenvalues*).

◂ Suppose v is a vector field with phase space U. Consider the mapping $v\colon U \to \mathbb{R}^n$, and suppose that a point 0 takes the role of the submanifold C. By the weak transversality theorem, a generic mapping v is transversal to C. This implies the nondegeneracy of the singular points of v. ▸

Theorem. *All the singular points of a generic vector field are hyperbolic.*

◂ Consider the one-jet extension of the mapping v from the phase space U to $\mathbb{R}^n$. The space $J^1(U, \mathbb{R}^n)$ consists of points of the form (x, y, A), where $x \in U$, $y \in \mathbb{R}^n$,