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**Advanced Lectures in Mathematics**

# **Geometry and Analysis (Vol. II)**

几何与分析 (第 II 卷)

Editor: Lizhen Ji



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*Jihe Yu Fenxi (Di II Juan)*

Editor: Lizhen Ji

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# ADVANCED LECTURES IN MATHEMATICS

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**Dedicated to Shing-Tung Yau on the occasion of  
his sixtieth birthday.**

## Preface

To celebrate the 60th birthday of Professor Shing-Tung Yau, a conference titled *Geometric Analysis: Present and Future* was held at Harvard University from August 27 to September 1, 2008.

The purpose of this conference is to summarize what has been achieved in the fields around geometric analysis in the past, to discuss the most recent results and to map out directions for the future. The title *Geometric Analysis* was interpreted very broadly and reflected the wide range of interests of Yau. It was also reflected in the topics of many talks at the conference.

There were 47 distinguished speakers and they are:

1. Robert Bartnik, Monash University, Australia.
2. Robert Bryant, Mathematical Sciences Research Institute, Berkeley, USA.
3. Jean-Pierre Bourguignon, Institut Hautes Etudes Scientifiques, France.
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13. Jiaying Hong, Fudan University, China.
14. Tristan Hubsch, Howard University, USA.
15. Gerhard Huisken, Albert-Einstein-Institut, Germany.
16. Blaine Lawson, State University of New York at Stony Brook, USA.
17. Jun Li, Stanford University, USA.
18. Peter Li, University of California, Irvine, USA.
19. Bong Lian, Brandeis University, USA.
20. Chang-Shou Lin, Taiwan University, China.

21. Fang-Hua Lin, New York University, USA.
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27. Yoichi Miyaoka, University of Tokyo, Japan.
28. Duong Phong, Columbia University, USA.
29. Wilfried Schmid, Harvard University, USA.
30. Richard Schoen, Stanford University, USA.
31. Leon Simon, Stanford University, USA.
32. Isadore Singer, Massachusetts Institute of Technology, USA.
33. Yum-Tong Siu, Harvard University, USA.
34. Joel Smoller, University of Michigan, USA.
35. James Sparks, Oxford University, UK.
36. Andrew Strominger, Harvard University, USA.
37. Cliff Taubes, Harvard University, USA.
38. Chuu-Lian Terng, University of California, Irvine, USA.
39. Karen Uhlenbeck, University of Texas, Austin, USA.
40. Cumrun Vafa, Harvard University, USA.
41. Chin-Lung Wang, Taiwan University, China.
42. Xujia Wang, Australian National University, Australia.
43. Ben Weinkove, University of California, San Diego, USA.
44. Edward Witten, Institute for Advanced Study, Princeton, USA.
45. Zhouping Xin, The Chinese University of Hong Kong, China. USA.
46. Eric Zaslow, Northwestern University, USA.
47. Xi-Ping Zhu, Sun Yat-Sen University, China.

There was also an open problem session in the evening of August 27, and a lunch forum *Women in Mathematics* on August 28. The conference banquet was held at the American Academy of Arts and Sciences in the evening of August 29.

The conference was a great success. We would like to thank all the speakers for their excellent talks and active participation in the conference. We would also like to take this opportunity to thank the following sponsors of this conference:

1. The National Science Foundation, USA.
2. The Chinese University of Hong Kong, China.
3. Department of Mathematics, The Chinese University of Hong Kong, China.

4. Department of Mathematics, Harvard University, USA.
5. Morningside Center of Mathematics, Chinese Academy of Sciences, China.
6. Journal of Differential Geometry, Lehigh University, USA.
7. Pacific Institute for the Mathematical Sciences.
8. Department of Mathematics, Massachusetts Institute of Technology, USA.
9. Department of Mathematics, Brandeis University.
10. Pure and Applied Mathematics Quarterly, USA.
11. Center of Mathematical Sciences, Zhejiang University, China.
12. Peter Viem Kwok, CITIC Resources Holdings Limited, Hong Kong, China.

The organizers of this conference consisted of the following:

1. Lizhen Ji, University of Michigan, USA.
2. Ka-Sing Lau, The Chinese University of Hong Kong, China.
3. Peter Li, University of California, Irvine, USA.
4. Kefeng Liu, University of California, Los Angeles, USA.
5. Wilfried Schmid, Harvard University, USA.
6. Rick Schoen, Stanford University, USA.
7. I.M. Singer, Massachusetts Institute of Technology, USA.
8. Clifford Taubes, Harvard University, USA.
9. Cumrun Vafa, Harvard University, USA.
10. Zhouping Xin, The Chinese University of Hong Kong, China.

Certainly, on behalf of the organizers, we would also like to thank the staff members of the mathematics department, Harvard University, for their generous help with this big and intense conference.

The current set of two volumes *Geometry and Analysis* is the proceedings of the conference and mainly consists of contributions of the speakers. Together with the following three volumes of *Handbook of Geometric Analysis* prepared in conjunction with the conference:

1. *Handbook of geometric analysis. Volume 1.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 7. International Press, Somerville, MA; Higher Education Press, Beijing, 2008. xii+676 pp.
2. *Handbook of geometric analysis. Volume 2.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 13. International Press, Somerville, MA; Higher Education Press, Beijing, 2010. x+432 pp.
3. *Handbook of geometric analysis. Volume 3.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 14. International Press, Somerville, MA; Higher Education Press, Beijing, 2010. x+472 pp.



We hope that they give a global perspective on the current status of geometric analysis and will be helpful in training the next generation of mathematicians. We would like to thank the authors of all these papers for their substantial contributions, and the referees for their generous help in the reviewing process.

These two volumes of *Geometry and Analysis* consist of 4 parts:

1. Summaries and commentaries on the work of Yau, which contains 20 short articles commenting on different aspects of Yau's work, lists of his papers and books, and a recent CV of him.
2. Differential geometry and differential equations, which contains 14 papers.
3. Mathematical physics, algebraic geometry and other topics, which also contains 14 papers.
4. Appendices, which contains a biography of Yau, and two survey papers by Yau on geometric analysis and Calabi-Yau manifolds.

Though geometric analysis has a long history, the decisive contributions of Yau since 1970s have made it an indispensable tool in many subjects such as differential geometry, topology, algebraic geometry, mathematical physics, etc, and hence have established it as one of the most important fields of modern mathematics. Yau's impacts are clearly visible in the papers of these two volumes, and we hope that these two volumes of *Geometry and Analysis* and the three volumes of the *Handbook of Geometric Analysis* will pay a proper tribute to him in a modest way.

According to the Chinese tradition, a person is one year old when he is born, and hence Yau turned 60 already in 2008. The number 60 and hence the age 60 is special in many cultures, especially in the Chinese culture. It is the smallest common multiple of 10 and 12, two important periods in the Chinese astronomy. Therefore, it is a new starting point (or a new cycle). A quick look at Yau's list of publications in Part 1 shows that Yau has not only maintained but increased his incredible output both in terms of quality and quantity.

On behalf of his friends, students and colleagues, we wish him continuing success and many productive, energetic years to come!

Lizhen Ji  
March 2010

# Contents

## Part 3 Mathematical Physics, Algebraic Geometry and Other Topics

### The Coherent-Constructible Correspondence

#### and Homological Mirror Symmetry for Toric Varieties

*Bohan Fang, Chiu-Chu Melissa Liu, David Treumann and Eric Zaslow.* 3

|     |                                                  |    |
|-----|--------------------------------------------------|----|
| 1   | Introduction .....                               | 3  |
| 1.1 | Outline .....                                    | 4  |
| 2   | Mirror symmetry for toric manifolds.....         | 4  |
| 2.1 | Hori-Vafa mirror .....                           | 4  |
| 2.2 | Categories in mirror symmetry .....              | 5  |
| 2.3 | Results to date.....                             | 8  |
| 3   | T-duality .....                                  | 9  |
| 3.1 | Moment polytope .....                            | 9  |
| 3.2 | Geometry of the open orbit .....                 | 10 |
| 3.3 | Statement of symplectic results .....            | 12 |
| 3.4 | T-dual of an equivariant line bundle.....        | 14 |
| 4   | Microlocalization.....                           | 16 |
| 4.1 | Algebraic preliminaries.....                     | 16 |
| 4.2 | The cast of categories .....                     | 17 |
| 4.3 | Fukaya-Oh theorem .....                          | 19 |
| 4.4 | Building the equivalence .....                   | 20 |
| 4.5 | Equivalence and the inverse functor .....        | 21 |
| 4.6 | Singular support and characteristic cycles ..... | 22 |
| 4.7 | Comments on technicalities .....                 | 23 |
| 4.8 | Statement of results.....                        | 25 |
| 5   | Coherent-constructible correspondence .....      | 25 |
| 6   | Examples .....                                   | 28 |
| 6.1 | Taking the mapping cone .....                    | 28 |
| 6.2 | Toric Fano surfaces .....                        | 29 |
| 6.3 | Hirzebruch surfaces .....                        | 29 |
|     | References .....                                 | 34 |

### Superspace: a Comfortably Vast Algebraic Variety

*T. Hübsch*..... 39

|     |                                   |    |
|-----|-----------------------------------|----|
| 1   | Introduction .....                | 39 |
| 1.1 | Basic ideas and definitions ..... | 40 |
| 1.2 | The traditional superspace .....  | 42 |

|                                                                 |                                                           |     |
|-----------------------------------------------------------------|-----------------------------------------------------------|-----|
| 2                                                               | Off-shell worldline supermultiplets .....                 | 44  |
| 2.1                                                             | Adinkraic supermultiplets .....                           | 45  |
| 2.2                                                             | Various hangings .....                                    | 46  |
| 2.3                                                             | Projected supermultiplets .....                           | 48  |
| 2.4                                                             | Supermultiplets <i>vs.</i> superfields .....              | 49  |
| 3                                                               | Superspace, by construction .....                         | 50  |
| 3.1                                                             | Superpartners of time .....                               | 50  |
| 3.2                                                             | A telescoping deformation structure .....                 | 55  |
| 3.3                                                             | Nontrivial superspace geometry .....                      | 59  |
| 3.4                                                             | Higher-dimensional spacetime .....                        | 62  |
| 4                                                               | The comfortably vast superspace .....                     | 63  |
|                                                                 | References .....                                          | 65  |
| <br><b>A Report on the Yau-Zaslow Formula</b>                   |                                                           |     |
|                                                                 | <i>Naichung Conan Leung</i> .....                         | 69  |
| 1                                                               | Yau-Zaslow formula and its generalizations .....          | 70  |
| 2                                                               | Yau-Zaslow approach .....                                 | 72  |
| 3                                                               | Matching method .....                                     | 72  |
| 4                                                               | Degeneration method .....                                 | 74  |
| 5                                                               | Calabi-Yau threefold method .....                         | 77  |
| 6                                                               | Conclusions .....                                         | 78  |
|                                                                 | References .....                                          | 79  |
| <br><b>Hermitian-Yang-Mills Connections on Kähler Manifolds</b> |                                                           |     |
|                                                                 | <i>Jun Li</i> .....                                       | 81  |
| 1                                                               | Introduction .....                                        | 81  |
| 1.1                                                             | Hermitian-Yang-Mills connections .....                    | 81  |
| 1.2                                                             | HYM connections lead to stable bundles .....              | 83  |
| 1.3                                                             | Stable bundles and their moduli spaces .....              | 85  |
| 1.4                                                             | Flat bundles and stable bundles on curves .....           | 86  |
| 2                                                               | Donaldson-Uhlenbeck-Yau theorem .....                     | 86  |
| 2.1                                                             | Donaldson's proof for algebraic surfaces .....            | 87  |
| 2.2                                                             | Uhlenbeck-Yau's proof for Kähler manifolds .....          | 88  |
| 3                                                               | Hermitian-Yang-Mills connections on curves .....          | 90  |
| 4                                                               | Hermitian-Yang-Mills connections on surfaces .....        | 92  |
| 4.1                                                             | Extending DUY correspondence .....                        | 92  |
| 4.2                                                             | Stable topology of the moduli spaces .....                | 94  |
| 4.3                                                             | Donaldson polynomial invariants .....                     | 95  |
| 5                                                               | HYM connections on high dimensional varieties .....       | 97  |
| 5.1                                                             | Extending the DUY correspondence in high dimensions ..... | 97  |
| 5.2                                                             | Donaldson-Thomas invariants .....                         | 98  |
| 6                                                               | Concluding remark .....                                   | 99  |
|                                                                 | References .....                                          | 99  |
| <br><b>Additivity and Relative Kodaira Dimensions</b>           |                                                           |     |
|                                                                 | <i>Tian-Jun Li and Weiyi Zhang</i> .....                  | 103 |
| 1                                                               | Introduction .....                                        | 103 |

|     |                                                                         |     |
|-----|-------------------------------------------------------------------------|-----|
| 2   | Kodaira Dimensions and fiber bundles .....                              | 104 |
| 2.1 | $\kappa^h$ for complex manifolds and $\kappa^t$ up to dimension 3 ..... | 105 |
| 2.2 | $\kappa^s$ for symplectic 4-manifolds .....                             | 107 |
| 2.3 | Additivity for a fiber bundle .....                                     | 109 |
| 3   | Embedded symplectic surfaces and relative Kod. dim. in dim. 4. .        | 112 |
| 3.1 | Embedded symplectic surfaces and maximality .....                       | 112 |
| 3.2 | The adjoint class $K_\omega + [F]$ .....                                | 115 |
| 3.3 | Existence and Uniqueness of relatively minimal model .....              | 122 |
| 3.4 | $\kappa^s(M, \omega, F)$ .....                                          | 124 |
| 4   | Relative Kod. dim. in dim. 2 and fibrations over a surface .....        | 127 |
| 4.1 | $\kappa^t(F, D)$ , Riemann-Hurwitz formula and Seifert fibrations ...   | 128 |
| 4.2 | Lefschetz fibrations .....                                              | 130 |
|     | References .....                                                        | 133 |

## Descendent Integrals and Tautological Rings of

### Moduli Spaces of Curves

|     |                                                               |     |
|-----|---------------------------------------------------------------|-----|
|     | <i>Kefeng Liu and Hao Xu</i> .....                            | 137 |
| 1   | Introduction .....                                            | 137 |
| 2   | Intersection numbers and the Witten-Kontsevich theorem .....  | 138 |
| 2.1 | Witten-Kontsevich theorem .....                               | 139 |
| 2.2 | Virasoro constraints .....                                    | 141 |
| 3   | The $n$ -point function .....                                 | 142 |
| 3.1 | A recursive formula of $n$ -point functions .....             | 142 |
| 3.2 | An effective recursion formulae of descendent integrals ..... | 145 |
| 4   | Hodge integrals .....                                         | 146 |
| 4.1 | Faber's algorithm .....                                       | 146 |
| 4.2 | Hodge integral formulae .....                                 | 147 |
| 5   | Higher Weil-Petersson volumes .....                           | 149 |
| 5.1 | Generalization of Mirzakhani's recursion formula .....        | 149 |
| 5.2 | Recursion formulae of higher Weil-Petersson volumes .....     | 151 |
| 6   | Faber's conjecture on tautological rings .....                | 152 |
| 6.1 | The Faber intersection number conjecture .....                | 153 |
| 6.2 | Relations with $n$ -point functions .....                     | 154 |
| 7   | Dimension of tautological rings .....                         | 155 |
| 7.1 | Ramanujan's mock theta functions .....                        | 156 |
| 7.2 | Asymptotics of tautological dimensions .....                  | 158 |
| 8   | Gromov-Witten invariants .....                                | 161 |
| 8.1 | Universal equations of Gromov-Witten invariants .....         | 162 |
| 8.2 | Some vanishing identities .....                               | 163 |
| 9   | Witten's $r$ -spin numbers .....                              | 164 |
| 9.1 | Generalized Witten's conjecture .....                         | 165 |
| 9.2 | An algorithm for computing Witten's $r$ -spin numbers .....   | 166 |
|     | References .....                                              | 168 |

## A General Voronoi Summation Formula for $GL(n, \mathbb{Z})$

|   |                                                    |     |
|---|----------------------------------------------------|-----|
|   | <i>Stephen D. Miller and Wilfried Schmid</i> ..... | 173 |
| 1 | Introduction .....                                 | 173 |

|   |                                      |     |
|---|--------------------------------------|-----|
| 2 | Automorphic Distributions.....       | 178 |
| 3 | Vanishing to infinite order .....    | 189 |
| 4 | Classical proof of the formula ..... | 206 |
| 5 | Adelic proof of the formula.....     | 211 |
|   | References.....                      | 223 |

## Geometry of Holomorphic Isometries and Related Maps between Bounded Domains

|     |                                                                                                                                                            |     |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
|     | <i>Ngaiming Mok</i> .....                                                                                                                                  | 225 |
| 1   | Examples of holomorphic isometries.....                                                                                                                    | 229 |
| 1.1 | Examples of equivariant embeddings into the<br>projective space.....                                                                                       | 229 |
| 1.2 | Non-standard holomorphic isometries of the Poincaré<br>disk into polydisks .....                                                                           | 231 |
| 1.3 | A non-standard holomorphic isometry of the Poincaré<br>disk into a Siegel upper half-plane.....                                                            | 232 |
| 1.4 | Examples of holomorphic isometries with arbitrary<br>normalizing constants $\lambda > 1$ .....                                                             | 232 |
| 2   | Analytic continuation of germs of holomorphic isometries.....                                                                                              | 234 |
| 2.1 | Analytic continuation of holomorphic isometries into the<br>projective space equipped with the Fubini-Study metric .....                                   | 234 |
| 2.2 | An extension and rigidity problem arising from<br>commutators of modular correspondences .....                                                             | 236 |
| 2.3 | Analytic continuation of holomorphic isometries up to<br>normalizing constants with respect to the Bergman<br>metric – extension beyond the boundary ..... | 240 |
| 2.4 | Canonically embeddable Bergman manifolds and<br>Bergman meromorphic compactifications .....                                                                | 246 |
| 3   | Holomorphic isometries of the Poincaré disk into<br>bounded symmetric domains .....                                                                        | 249 |
| 3.1 | Structural equations on the norm of the second<br>fundamental form and asymptotic vanishing order.....                                                     | 249 |
| 3.2 | Holomorphic isometries of the Poincaré disk into<br>polydisks: structural results.....                                                                     | 250 |
| 3.3 | Calculated examples on the norm of the second<br>fundamental form .....                                                                                    | 251 |
| 3.4 | Holomorphic isometries of the Poincaré disk into<br>polydisks: uniqueness results .....                                                                    | 253 |
| 3.5 | Asymptotic total geodesy and applications .....                                                                                                            | 254 |
| 4   | Measure-preserving algebraic correspondences on irreducible<br>bounded symmetric domains .....                                                             | 255 |
| 4.1 | Statements of results.....                                                                                                                                 | 255 |
| 4.2 | Extension results on strictly pseudoconvex<br>algebraic hypersurfaces .....                                                                                | 256 |
| 4.3 | Alexander-type extension results in the higher-rank case ....                                                                                              | 257 |
| 4.4 | Total geodesy of germs of measure-preserving holomorphic                                                                                                   |     |

|                                                                                                                                                           |     |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| map from an irreducible bounded symmetric domain of dimension $\geq 2$ into its Cartesian products .....                                                  | 259 |
| 5 Open problems .....                                                                                                                                     | 261 |
| 5.1 On the structure of the space of holomorphic isometries of the Poincaré disk into polydisks .....                                                     | 261 |
| 5.2 On the second fundamental form and asymptotic behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains .....            | 264 |
| 5.3 On germs of holomorphic maps preserving invariant $(p, p)$ -forms .....                                                                               | 266 |
| References .....                                                                                                                                          | 267 |
| <b>Abundance Conjecture</b>                                                                                                                               |     |
| <i>Yum-Tong Siu</i> .....                                                                                                                                 | 271 |
| 0 Introduction .....                                                                                                                                      | 271 |
| 1 Curvature current and dichotomy .....                                                                                                                   | 275 |
| 2 Gelfond-Schneider's technique of algebraic values of solutions of algebraically defined differential equations .....                                    | 279 |
| 3 Final step of the case of zero numerical Kodaira dimension .....                                                                                        | 287 |
| 4 Numerically trivial foliations and fibrations for canonical line bundle .....                                                                           | 289 |
| 5 Curvature of zeroth direct image of relative canonical and pluricanonical bundle .....                                                                  | 291 |
| 6 Strict positivity of direct image of relative pluricanonical bundle along numerically trivial fibers in the base of numerically trivial fibration ..... | 302 |
| 7 Technique of Nevanlinna's first main theorem for proof of compactness of leaves of foliation .....                                                      | 306 |
| References .....                                                                                                                                          | 315 |
| <b>Sasaki-Einstein Geometry</b>                                                                                                                           |     |
| <i>James Sparks</i> .....                                                                                                                                 | 319 |
| 1 Sasakian geometry .....                                                                                                                                 | 319 |
| 2 Constructions of Sasaki-Einstein manifolds .....                                                                                                        | 321 |
| 3 Obstructions .....                                                                                                                                      | 324 |
| 4 Sasaki-Einstein manifolds in string theory .....                                                                                                        | 326 |
| References .....                                                                                                                                          | 327 |
| <b>A Simple Proof of the Chiral Gravity Conjecture</b>                                                                                                    |     |
| <i>Andrew Strominger</i> .....                                                                                                                            | 329 |
| <b>Geometry of Grand Unification</b>                                                                                                                      |     |
| <i>Cumrun Vafa</i> .....                                                                                                                                  | 335 |
| 1 Introduction .....                                                                                                                                      | 335 |
| 2 Standard model and gauge symmetry breaking .....                                                                                                        | 336 |
| 3 Flavors and hierarchy .....                                                                                                                             | 337 |
| 4 Unification of gauge groups .....                                                                                                                       | 338 |
| 5 String theory, forces, matter, and interactions .....                                                                                                   | 339 |

|     |                                        |     |
|-----|----------------------------------------|-----|
| 6   | F-theory vacua .....                   | 340 |
| 6.1 | Matter fields .....                    | 340 |
| 6.2 | Yukawa couplings .....                 | 342 |
| 7   | Applications to particle physics ..... | 342 |
| 7.1 | E-type singularity .....               | 343 |
| 7.2 | Flavor hierarchy .....                 | 343 |
| 7.3 | Breaking to the standard model .....   | 344 |
| 8   | Further issues .....                   | 345 |
|     | References .....                       | 345 |

## Quantum Invariance Under Flop Transitions

|     |                                                                                                                |     |
|-----|----------------------------------------------------------------------------------------------------------------|-----|
|     | <i>Chin-Lung Wang</i> .....                                                                                    | 347 |
| 1   | Introduction .....                                                                                             | 347 |
| 2   | Ordinary flops: Genus zero theory .....                                                                        | 350 |
| 2.1 | The canonical correspondence .....                                                                             | 350 |
| 2.2 | The case of simple flops [9] .....                                                                             | 351 |
| 2.3 | The topological defect .....                                                                                   | 352 |
| 2.4 | The extremal functions .....                                                                                   | 353 |
| 2.5 | Degeneration analysis .....                                                                                    | 355 |
| 2.6 | The local models .....                                                                                         | 356 |
| 3   | Calabi-Yau flops .....                                                                                         | 358 |
| 3.1 | The basic setup .....                                                                                          | 358 |
| 3.2 | $I, P, J$ and their degrees .....                                                                              | 358 |
| 3.3 | The CY condition and the mirror map .....                                                                      | 360 |
| 3.4 | Example: Flops of type $(P^1, \mathcal{O} \oplus \mathcal{O}(-7)), \mathcal{O}(3) \oplus \mathcal{O}(2)$ ..... | 361 |
| 3.5 | Proof of the main result in the example .....                                                                  | 365 |
|     | References .....                                                                                               | 369 |

## The Problem Of Gauge Theory

|   |                                                         |     |
|---|---------------------------------------------------------|-----|
|   | <i>Edward Witten</i> .....                              | 371 |
| 1 | Yang-Mills equations .....                              | 371 |
| 2 | Classical phase space .....                             | 373 |
| 3 | Quantization .....                                      | 376 |
| 4 | Nonperturbative approach .....                          | 379 |
| 5 | Breaking of conformal invariance and the mass gap ..... | 381 |
|   | References .....                                        | 382 |

## Part 4 Appendices

### Shing-Tung Yau, a Manifold Man of Mathematics

|   |                                            |     |
|---|--------------------------------------------|-----|
|   | <i>Lizhen Ji and Kefeng Liu</i> .....      | 385 |
| 1 | Childhood and early school education ..... | 386 |
| 2 | Middle school and college .....            | 387 |
| 3 | Graduate school .....                      | 390 |
| 4 | Professional career .....                  | 392 |
| 5 | Major contributions to mathematics .....   | 395 |
| 6 | Visits to China .....                      | 404 |

|    |                                                   |     |
|----|---------------------------------------------------|-----|
| 7  | Research centers and mathematics institutes ..... | 405 |
| 8  | ICCM .....                                        | 408 |
| 9  | Conferences and popular mathematics programs..... | 409 |
| 10 | Mathematics and Chinese literature.....           | 410 |
| 11 | Family, friends and students .....                | 410 |
| 12 | Summary .....                                     | 415 |
|    | References.....                                   | 416 |

## Perspectives on Geometric Analysis

|     |                                                                                              |     |
|-----|----------------------------------------------------------------------------------------------|-----|
|     | <i>Shing-Tung Yau</i> .....                                                                  | 417 |
| 1   | History and contributors of the subject.....                                                 | 419 |
| 1.1 | Founding fathers of the subject .....                                                        | 419 |
| 1.2 | Modern Contributors .....                                                                    | 421 |
| 2   | Construction of functions in geometry.....                                                   | 422 |
| 2.1 | Polynomials from ambient space.....                                                          | 423 |
| 2.2 | Geometric construction of functions .....                                                    | 426 |
| 2.3 | Functions and tensors defined by linear<br>differential equations .....                      | 430 |
| 3   | Mappings between manifolds and rigidity<br>of geometric structures .....                     | 446 |
| 3.1 | Embedding .....                                                                              | 446 |
| 3.2 | Rigidity of harmonic maps with negative curvature.....                                       | 449 |
| 3.3 | Holomorphic maps .....                                                                       | 451 |
| 3.4 | Harmonic maps from two dimensional surfaces and<br>pseudoholomorphic curves .....            | 452 |
| 3.5 | Morse theory for maps and topological applications .....                                     | 453 |
| 3.6 | Wave maps .....                                                                              | 454 |
| 3.7 | Integrable system .....                                                                      | 454 |
| 3.8 | Regularity theory .....                                                                      | 455 |
| 4   | Submanifolds defined by variational principles.....                                          | 455 |
| 4.1 | Teichmüller space .....                                                                      | 455 |
| 4.2 | Classical minimal surfaces in Euclidean space .....                                          | 456 |
| 4.3 | Douglas-Morrey solution, embeddedness and<br>application to topology of three manifolds..... | 457 |
| 4.4 | Surfaces related to classical relativity .....                                               | 458 |
| 4.5 | Higher dimensional minimal subvarieties .....                                                | 459 |
| 4.6 | Geometric flows .....                                                                        | 462 |
| 5   | Construction of geometric structures on bundles and manifolds...                             | 463 |
| 5.1 | Geometric structures with noncompact holonomy group.....                                     | 464 |
| 5.2 | Uniformization for three manifolds .....                                                     | 466 |
| 5.3 | Four manifolds.....                                                                          | 469 |
| 5.4 | Special connections on bundles.....                                                          | 470 |
| 5.5 | Symplectic structures .....                                                                  | 471 |
| 5.6 | Kähler structure .....                                                                       | 474 |
| 5.7 | Manifolds with special holonomy group .....                                                  | 480 |
| 5.8 | Geometric structures by reduction .....                                                      | 480 |



|      |                                                                             |     |
|------|-----------------------------------------------------------------------------|-----|
| 5.9  | Obstruction for existence of Einstein metrics<br>on general manifolds ..... | 481 |
| 5.10 | Metric Cobordism .....                                                      | 481 |
|      | References .....                                                            | 482 |

## A Survey of Calabi-Yau Manifolds

|     |                                                                                  |     |
|-----|----------------------------------------------------------------------------------|-----|
|     | <i>Shing-Tung Yau</i> .....                                                      | 521 |
| 1   | Introduction .....                                                               | 521 |
| 2   | General Constructions of Complete Ricci-Flat Metrics<br>in Kähler Geometry ..... | 521 |
| 2.1 | The Ricci tensor of Calabi-Yau manifolds .....                                   | 521 |
| 2.2 | The Calabi conjecture .....                                                      | 522 |
| 2.3 | Yau's theorem .....                                                              | 522 |
| 2.4 | Calabi-Yau manifolds and Calabi-Yau metrics .....                                | 523 |
| 2.5 | Examples of compact Calabi-Yau manifolds .....                                   | 524 |
| 2.6 | Noncompact Calabi-Yau manifolds .....                                            | 525 |
| 2.7 | Calabi-Yau cones: Sasaki-Einstein manifolds .....                                | 526 |
| 2.8 | The balanced condition on Calabi-Yau metrics .....                               | 527 |
| 3   | Moduli and Arithmetic of Calabi-Yau Manifolds .....                              | 528 |
| 3.1 | Moduli of K3 surfaces .....                                                      | 528 |
| 3.2 | Moduli of high dimensional Calabi-Yau manifolds .....                            | 529 |
| 3.3 | The modularity of Calabi-Yau threefolds over $\mathbb{Q}$ .....                  | 530 |
| 4   | Calabi-Yau Manifolds in Physics .....                                            | 531 |
| 4.1 | Calabi-Yau manifolds in string theory .....                                      | 531 |
| 4.2 | Calabi-Yau manifolds and mirror symmetry .....                                   | 532 |
| 4.3 | Mathematics inspired by mirror symmetry .....                                    | 533 |
| 5   | Invariants of Calabi-Yau Manifolds .....                                         | 534 |
| 5.1 | Gromov-Witten Invariants .....                                                   | 534 |
| 5.2 | Counting formulas .....                                                          | 534 |
| 5.3 | Proofs of counting formulas for Calabi-Yau threefolds .....                      | 535 |
| 5.4 | Integrability of mirror map and arithmetic applications .....                    | 536 |
| 5.5 | Donaldson-Thomas invariants .....                                                | 537 |
| 5.6 | Stable bundles and sheaves .....                                                 | 538 |
| 5.7 | Yau-Zaslow formula for K3 surfaces .....                                         | 539 |
| 5.8 | Chern-Simons knot invariants, open strings and<br>string dualities .....         | 540 |
| 6   | Homological Mirror Symmetry .....                                                | 541 |
| 7   | SYZ geometric interpretation of mirror symmetry .....                            | 542 |
| 7.1 | Special Lagrangian submanifolds in Calabi-Yau manifolds ...                      | 542 |
| 7.2 | The SYZ conjecture - SYZ transformation .....                                    | 543 |
| 7.3 | Special Lagrangian geometry .....                                                | 543 |
| 7.4 | Special Lagrangian fibrations .....                                              | 544 |
| 7.5 | The SYZ transformation .....                                                     | 545 |
| 7.6 | The SYZ conjecture and tropical geometry .....                                   | 545 |
| 8   | Geometries Related to Calabi-Yau Manifolds .....                                 | 546 |
| 8.1 | Non-Kähler Calabi-Yau manifolds .....                                            | 546 |