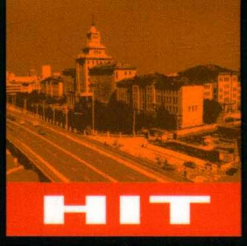
The Theory of the Riemann Zeta-function

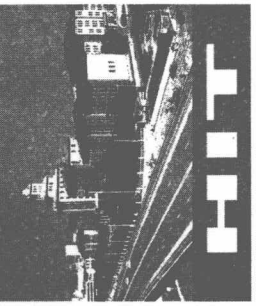


国外优秀数学著作
原 版 系 列

黎曼ZETA函数的理论

[英] E.C. 蒂奇玛什（Titchmarsh,Edward Charles）著

哈爾濱工業大學出版社
HARBIN INSTITUTE OF TECHNOLOGY PRESS



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● [英] E.C. 蒂奇玛什（Titchmarsh,Edward Charles） 著

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内容简介

本书主要继承了作者本人的剑桥小册子 *The Zeta-function of Riemann* 的内容. 本书内容主要包括:$\zeta(s)$ 函数,狄利克雷级数与 $\zeta(s)$ 函数的关系,$\zeta(s)$ 函数的分析特点,函数方程,近似公式,$\zeta(s)$ 函数在临界带的次序。

本书适合数学专业人士和数学爱好者参考阅读。

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PREFACE TO THE SECOND EDITION

Since the first edition was written, a vast amount of further work has been done. This has been covered by the end-of-chapter notes. In most instances, restrictions on space have prohibited the inclusion of full proofs, but I have tried to give an indication of the methods used wherever possible. (Proofs of quite a few of the recent results described in the end of chapter notes may be found in the book by Ivic [**3**].) I have also corrected a number of minor errors, and made a few other small improvements to the text. A considerable number of recent references have been added.

In preparing this work I have had help from Professors J. B. Conrey, P. D. T. A. Elliott, A. Ghosh, S. M. Gonek, H. L. Montgomery and S. J. Patterson. It is a pleasure to record my thanks to them.

D. R. H. B.
OXFORD
1986

PREFACE TO THE FIRST EDITION

This book is a successor to my Cambridge Tract *The Zeta-Function of Riemann*, 1930, which is now out of print and out of date. It seems no longer practicable to give an account of the subject in such a small space as a Cambridge Tract, so that the present work, though on exactly the same lines as the previous one, is on a much larger scale. As before, I do not discuss general prime-number theory, though it has been convenient to include some theorems on primes.

Most of this book was compiled in the 1930's, when I was still researching on the subject. It has been brought partly up to date by including some of the work of A. Selberg and of Vinogradov, though a great deal of recent work is scantily represented.

The manuscript has been read by Dr. S. H. Min and by Prof. D. B. Sears, and my best thanks are due to them for correcting a large number of mistakes. I must also thank Prof. F. V. Atkinson and Dr. T. M. Fleet for their kind assistance in reading the proof-sheets.

E. C. T.
OXFORD
1951

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Chapter I

THE FUNCTION $\zeta(s)$ AND THE DIRICHLET SERIES RELATED TO IT

1.1. Definition of $\zeta(s)$. The Riemann zeta-function $\zeta(s)$ has its origin in the identity expressed by the two formulae

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1.1}$$

where n runs through all integers, and

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.1.2}$$

where p runs through all primes. Either of these may be taken as the definition of $\zeta(s)$; s is a complex variable, $s = \sigma + it$. The Dirichlet series (1.1.1) is convergent for $\sigma > 1$, and uniformly convergent in any finite region in which $\sigma \geqslant 1 + \delta$, $\delta > 0$. It therefore defines an analytic function $\zeta(s)$, regular for $\sigma > 1$.

The infinite product is also absolutely convergent for $\sigma > 1$; for so is

$$\sum_{p} \left|\frac{1}{p^s}\right| = \sum_{p} \frac{1}{p^\sigma}$$

this being merely a selection of terms from the series $\sum n^{-\sigma}$. If we expand the factor involving p in powers of p^{-s}, we obtain

$$\prod_{p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

On multiplying formally, we obtain the series (1.1.1), since each integer n can be expressed as a product of prime-powers p^m in just one way. The identity of (1.1.1) and (1.1.2) is thus an analytic equivalent of the theorem that the expression of an integer in prime factors is unique.

A rigorous proof is easily constructed by taking first a finite number of factors. Since we can multiply a finite number of absolutely convergent series, we have

$$\prod_{p\leqslant P}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)=1+\frac{1}{n_1^s}+\frac{1}{n_2^s}+\cdots$$

where n_1, n_2, $\cdots$, are those integers none of whose prime factors exceed P. Since all integers up to P are of this form, it follows that, if $\zeta(s)$ is defined by (1.1.1)

$$\left|\zeta(s)-\prod_{p\leqslant P}\left(1-\frac{1}{p^s}\right)^{-1}\right|=\left|\zeta(s)-1-\frac{1}{n_1^s}-\frac{1}{n_2^s}-\cdots\right|$$

$$\leqslant\frac{1}{(P+1)^\sigma}+\frac{1}{(P+2)^\sigma}+\cdots$$

This tends to 0 as $P\to\infty$, if $\sigma>1$; and (1.1.2) follows.

This fundamental identity is due to Euler, and (1.1.2) is known as Euler's product. But Euler considered it for particular values of s only, and it was Riemann who first considered $\zeta(s)$ as an analytic function of a complex variable.

Since a convergent infinite product of non-zero factors is not zero, we deduce that $\zeta(s)$ has no zeros for $\sigma>1$. This may be proved directly as follows. We have for $\sigma>1$

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\cdots\left(1-\frac{1}{P^s}\right)\zeta(s)=1+\frac{1}{m_1^s}+\frac{1}{m_2^s}+\cdots$$

where m_1, m_2, $\cdots$, are the integers all of whose prime factors exceed P. Hence

$$\left|\left(1-\frac{1}{2^s}\right)\cdots\left(1-\frac{1}{P^s}\right)\zeta(s)\right|\geqslant 1-\frac{1}{(P+1)^\sigma}-\frac{1}{(P+2)^\sigma}-\cdots>0$$

if P is large enough. Hence $|\zeta(s)|>0$.

The importance of $\zeta(s)$ in the theory of prime numbers lies in the fact that it combines two expressions, one of which contains the primes explicitly, while the other does not. The theory of primes is largely concerned with the function $\pi(x)$, the number of primes not exceeding x. We can transform (1.1.2) into a relation between $\zeta(s)$ and $\pi(x)$; for if $\sigma>1$

$$\log\zeta(s)=-\sum_p\log\left(1-\frac{1}{p^s}\right)=-\sum_{n=2}^{\infty}\{\pi(n)-\pi(n-1)\}\log\left(1-\frac{1}{n^s}\right)$$

$$=-\sum_{n=2}^{\infty}\pi(n)\left\{\log\left(1-\frac{1}{n^s}\right)-\log\left(1-\frac{1}{(n+1)^s}\right)\right\}$$

$$=\sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\mathrm{d}x=s\int_2^{\infty}\frac{\pi(x)}{x(x^s-1)}\mathrm{d}x \qquad (1.1.3)$$

The rearrangement of the series is justified since $\pi(n)\leqslant n$ and

$$\log(1-n^{-s})=O(n^{-\sigma})$$

Again

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right)$$

and on carrying out the multiplication we obtain

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad (\sigma > 1) \tag{1.1.4}$$

where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k different primes, and $\mu(n) = 0$ if n contains any factor to a power higher than the first. The process is easily justified as in the case of $\zeta(s)$.

The function $\mu(n)$ is known as the Möbius function. It has the property

$$\sum_{d|q} \mu(d) = 1 (q = 1) \quad 0 (q > 1) \tag{1.1.5}$$

where $d|q$ means that d is a divisor of q. This follows from the identity

$$1 = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d|q} \mu(d)$$

It also gives the Möbius inversion formula

$$g(q) = \sum_{d|q} f(d) \tag{1.1.6}$$

$$f(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right) g(d) \tag{1.1.7}$$

connecting two functions $f(n)$, $g(n)$ defined for integral n. If f is given and g defined by (1.1.6), the right-hand side of (1.1.7) is

$$\sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{r|d} f(r)$$

The coefficient of $f(q)$ is $\mu(1) = 1$. If $r < q$, then $d = kr$, where $k|q/r$. Hence the coefficient of $f(r)$ is

$$\sum_{k|q/r} \mu\left(\frac{q}{kr}\right) = \sum_{k'q/r} \mu(k') = 0$$

by (1.1.5). This proves (1.1.7). Conversely, if g is given, and f is defined by (1.1.7), then the right-hand side of (1.1.6) is

$$\sum_{d|q} \sum_{r|d} \mu\left(\frac{d}{r}\right) g(r)$$

and this is $g(q)$, by a similar argument. The formula may also be derived formally from the obviously equivalent relations

$$F(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad F(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

where

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Again, on taking logarithms and differentiating (1.1.2), we obtain, for $\sigma > 1$

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-2} \\ &= -\sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \\ &= -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \end{aligned} \tag{1.1.8}$$

where $\Lambda(n) = \log p$ if n is p or a power of p, and otherwise $\Lambda(n) = 0$. On integrating we obtain

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s} \quad (\sigma > 1) \tag{1.1.9}$$

where $\Lambda_1(n) = \Lambda(n)/\log n$, and the value of $\log \zeta(s)$ is that which tends to 0 as $\sigma \to \infty$, for any fixed t.

1.2. Various Dirichlet series connected with $\zeta(s)$. In the first place

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (\sigma > 1) \tag{1.2.1}$$

where $d(n)$ denotes the number of divisors of n (including 1 and n itself). For

$$\zeta^2(s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{\nu=1}^{\infty} \frac{1}{\nu^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mu\nu=n} 1$$

and the number of terms in the last sum is $d(n)$. And generally

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1) \tag{1.2.2}$$

where $k = 2, 3, 4, \cdots$, and $d_k(n)$ denotes the number of ways of expressing n as a product of k factors, expressions with the same factors in a different order being counted as different. For

$$\zeta^k(s) = \sum_{\nu_1=1}^{\infty} \frac{1}{\nu_1^s} \cdots \sum_{\nu_k=1}^{\infty} \frac{1}{\nu_k^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu_1 \cdots \nu_k = n} 1$$

and the last sum is $d_k(n)$.

Since we have also

$$\zeta^2(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-2} = \prod_p \left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\cdots\right) \tag{1.2.3}$$

on comparing the coefficients in (1.2.1) and (1.2.3) we verify the elementary formula

$$d(n) = (m_1+1)\cdots(m_r+1) \tag{1.2.4}$$

for the number of divisors of

$$n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} \tag{1.2.5}$$

Similarly from (1.2.2)

$$d_k(n) = \frac{(k+m_1-1)!}{m_1!\ (k-1)!} \cdots \frac{(k+m_r-1)!}{m_r!\ (k-1)!} \tag{1.2.6}$$

We next note the expansions

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \quad (\sigma > 1) \tag{1.2.7}$$

where $\mu(n)$ is the coefficient in (1.1.4)

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s} \quad (\sigma > 1) \tag{1.2.8}$$

where $\nu(n)$ is the number of different prime factors of n

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} \quad (\sigma > 1) \tag{1.2.9}$$

and

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\{d(n)\}^2}{n^s} \quad (\sigma > 1) \tag{1.2.10}$$

To prove (1.2.7), we have

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \prod_p \left(1 + \frac{1}{p^s}\right)$$

and this differs from the formula for $1/\zeta(s)$ only in the fact that the signs are all positive. The result is therefore clear. To prove (1.2.8), we have

$$\begin{aligned} \frac{\zeta^2(s)}{\zeta(2s)} &= \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^2} = \prod_p \frac{1+p^{-s}}{1-p^{-s}} \\ &= \prod_p (1 + 2p^{-s} + 2p^{-2s} + \cdots) \end{aligned}$$

and the result follows. To prove (1.2.9)

$$\begin{aligned} \frac{\zeta^3(s)}{\zeta(2s)} &= \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^3} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2} \\ &= \prod_p \{(1+p^{-s})(1+2p^{-s}+3p^{-2s}+\cdots)\} \\ &= \prod_p \{1 + 3p^{-s} + \cdots + (2m+1)p^{-ms} + \cdots\} \end{aligned}$$

and the result follows, since, if n is (1.2.5)

$$d(n^2) = (2m_1+1)\cdots(2m_r+1)$$

Similarly

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^4} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^3}$$

$$= \prod_p \{(1+p^{-s})\{1+3p^{-s}+\cdots+\frac{1}{2}(m+1)(m+2)p^{-ms}+\cdots\}$$

$$= \prod_p \{1+4p^{-s}+\cdots+(m+1)^2p^{-ms}+\cdots\}$$

and (1.2.10) follows.

Other formulae are

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad (\sigma>1) \tag{1.2.11}$$

where $\lambda(n) = (-1)^r$ if n has r prime factors, a factor of degree k being counted k times

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \quad (\sigma>2) \tag{1.2.12}$$

where $\phi(n)$ is the number of numbers less than n and prime to n; and

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (\sigma>2) \tag{1.2.13}$$

where $a(n)$ is the greatest odd divisor of n. Of these, (1.2.11) follows at once from

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}}\right) = \prod_p \left(\frac{1}{1+p^{-s}}\right) = \prod_p (1-p^{-s}+p^{-2s}-\cdots)$$

Also

$$\frac{\zeta(s-1)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{1-s}}\right) = \prod_p \left\{\left(1-\frac{1}{p^s}\right)\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\cdots\right)\right\}$$

$$= \prod_p \left\{1+\left(1-\frac{1}{p}\right)\left(\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\cdots\right)\right\}$$

and (1.2.12) follows, since, if $n = p_1^{m_1}\cdots p_r^{m_r}$

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

Finally

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \frac{1-2^{1-s}}{1-2^{-s}} \prod_p \frac{1}{1-p^{1-s}}$$

$$= \frac{1}{1-2^{-s}} \times \frac{1}{1-3^{1-s}} \times \frac{1}{1-5^{1-s}} \times \cdots$$

$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{3}{3^s}+\frac{3^2}{3^{2s}}+\cdots\right)\cdots$$

and (1.2.13) follows.

Many of these formulae are, of course, simply particular cases of the general formula

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_p\left\{1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\right\}$$

where $f(n)$ is a multiplicative function, i. e. is such that, if $n=p_1^{m_1}p_2^{m_2}\cdots$, then

$$f(n)=f(p_1^{m_1})f(p_2^{m_2})\cdots$$

Again, let $f_k(n)$ denote the number of representations of n as a product of k factors, each greater than unity when $n>1$, the order of the factors being essential. Then clearly

$$\sum_{n=2}^{\infty}\frac{f_k(n)}{n^2}=\{\zeta(s)-1\}^k\quad(\sigma>1)\tag{1.2.14}$$

Let $f(n)$ be the number of representations of n as a product of factors greater than unity, representations with factors in a different order being considered as distinct; and let $f(1)=1$. Then

$$f(n)=\sum_{k=1}^{\infty}f_k(n)$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{f(n)}{n^s}&=1+\sum_{k=1}^{\infty}\{\zeta(s)-1\}^k\\&=1+\frac{\zeta(s)-1}{1-\{\zeta(s)-1\}}\\&=\frac{1}{2-\zeta(s)}\end{aligned}\tag{1.2.15}$$

It is easily seen that $\zeta(s)=2$ for $s=\alpha$, where α is a real number greater than 1; and $|\zeta(s)|<2$ for $\sigma>\alpha$, so that (1.2.15) holds for $\sigma>\alpha$.

1.3. Sums involving $\sigma_a(n)$. Let $\sigma_a(n)$ denote the sum of the ath powers of the divisors of n. Then

$$\zeta(s)\zeta(s-a)=\sum_{n=1}^{\infty}\frac{1}{\mu^s}\sum_{\nu=1}^{\infty}\frac{\nu^a}{\nu^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\mu\nu=n}\nu^a$$

i. e.

$$\zeta(s)\zeta(s-a)=\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s}\quad(\sigma>1,\ \sigma>\mathbf{R}(a)+1)\tag{1.3.1}$$

Since the left-hand side is, if $a\neq0$

$$\begin{aligned}&\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\left(1+\frac{p^a}{p^s}+\frac{p^{2a}}{p^{2s}}+\cdots\right)\\&=\prod_p\left(1+\frac{1+p^a}{p^s}+\frac{1+p^a+p^{2a}}{p^s}+\cdots\right)\\&=\prod_p\left(1+\frac{1-p^{2a}}{1-p^a}\frac{1}{p^s}+\cdots\right)\end{aligned}$$

we have

$$\sigma_a(n) = \frac{1-p_1^{(m_1+1)a}}{1-p_1^a} \cdot \cdots \cdot \frac{1-p_r^{(m_r+1)a}}{1-p_r^a} \tag{1.3.2}$$

if n is (1.2.5), as is also obvious from elementary considerations.

The formula[①]

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \tag{1.3.3}$$

is valid for $\sigma > \max\{1, \mathbf{R}(a)+1, \mathbf{R}(b)+1, \mathbf{R}(a+b)+1\}$. The left-hand side is equal to

$$\prod_p \frac{1-p^{-2s+a+b}}{(1-p^{-s})(1-p^{-s+a})(1-p^{-s+b})(1-p^{-s+a+b})}$$

Putting $p^{-s}=z$, the partial-fraction formula gives

$$\begin{aligned}
&\frac{1-p^{a+b}z^2}{(1-z)(1-p^a z)(1-p^b z)(1-p^{a+b}z)} \\
&= \frac{1}{(1-p^a)(1-p^b)}\left\{\frac{1}{1-z} - \frac{p^a}{1-p^a z} - \frac{p^b}{1-p^b z} + \frac{p^{a+b}}{1-p^{a+b}z}\right\} \\
&= \frac{1}{(1-p^a)(1-p^b)} \sum_{m=0}^{\infty} (1-p^{(m+1)a} - p^{(m+1)b} + p^{(m+1)(a+b)}) z^m \\
&= \frac{1}{(1-p^a)(1-p^b)} \sum_{m=0}^{\infty} (1-p^{(m+1)a})(1-p^{(m+1)b}) z^m
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \\
&= \prod_p \sum_{m=0}^{\infty} \frac{1-p^{(m+1)a}}{1-p^a} \times \frac{1-p^{(m+1)b}}{1-p^b} \times \frac{1}{p^{ms}}
\end{aligned}$$

and the result follows from (1.3.2). If $a=b=0$, (1.3.3) reduces to (1.2.10).

Similar formulae involving $\sigma_a^{(q)}(n)$, the sum of the ath powers of those divisors of n which are qth powers of integers, have been given by Crum (1).

1.4. It is also easily seen that, if $f(n)$ is multiplicative, and

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is a product of zeta-functions such as occurs in the above formulae, and k is a given positive integer, then

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① Ramanujan (2), B. M. Wilson (1).

$$\sum_{n=1}^{\infty} \frac{f(kn)}{n^s}$$

can also be summed. An example will illustrate this point. The function $\sigma_a(n)$ is 'multiplicative', i. e. if m is prime to n

$$\sigma_a(mn) = \sigma_a(m)\sigma_a(n)$$

Hence

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)}{p^{ms}}$$

and, if $k = \prod p^l$

$$\sum_{n=1}^{\infty} \frac{\sigma_a(kn)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}}$$

Hence

$$\sum_{n=1}^{\infty} \frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a) \prod_{p|k} \left\{ \sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}} \Big/ \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)}{p^{ms}} \right\}$$

Now if $a \neq 0$

$$\sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}} = \sum_{m=0}^{\infty} \frac{1-p^{(l+m+1)a}}{(1-p^a)p^{ms}} = \frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{(1-p^a)(1-p^{-s})(1-p^{a-s})}$$

Hence

$$\sum_{n=0}^{\infty} \frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a) \prod_{p|k} \frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{1-p^a} \tag{1.4.1}$$

Making $a \neq 0$

$$\sum_{n=0}^{\infty} \frac{d(kn)}{n^s} = \zeta^2(s) \prod_{p|k} (l+1-lp^{-s}) \tag{1.4.2}$$

1.5. Ramanujan's sums[①]. Let

$$c_k(n) = \sum_h e^{-2nh\pi i/k} = \sum_h \cos \frac{2nh\pi}{k} \tag{1.5.1}$$

where h runs through all positive integers less than and prime to k. Many formulae involving these sums were proved by Ramanujan.

We shall first prove that

$$c_k(n) = \sum_{d|k,d|n} \mu\left(\frac{k}{d}\right) d \tag{1.5.2}$$

The sum

$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-2nm\pi i/k}$$

① Ramanujan (3), Hardy (5).

is equal to k if $k \mid n$ and 0 otherwise. Denoting by (r, d) the highest common factor of r and d, so that $(r, d) = 1$ means that r is prime to d

$$\sum_{d \mid k} c_d(n) = \sum_{d \mid k} \sum_{(r,d)=1, r<d} e^{-2nr\pi i/d} = \eta_k(n)$$

Hence by the inversion formula of Möbius (1.1.7)

$$c_k(n) = \sum_{d \mid k} \mu\left(\frac{k}{d}\right) \eta_d(n)$$

and (1.5.2) follows. In particular

$$c_k(1) = \mu(k) \tag{1.5.3}$$

The result can also be written

$$c_k(n) = \sum_{dr=k, d \mid n} \mu(r) d$$

Hence

$$\frac{c_k(n)}{k^s} = \sum_{dr=k, d \mid n} \frac{\mu(r)}{r^s} d^{1-s}$$

Summing with respect to k, we remove the restriction on r, which now assumes all positive integral values. Hence[①]

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{r, d \mid n} \frac{\mu(r)}{r^s} d^{1-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \tag{1.5.4}$$

the series being absolutely convergent for $\sigma > 1$ since $|c_k(n)| \leqslant \sigma_1(n)$, by (1.5.2).

We have also

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{c_k(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d \mid k, d \mid n} \mu\left(\frac{k}{d}\right) d \\ &= \sum_{d \mid k} \mu\left(\frac{k}{d}\right) d \sum_{m=1}^{\infty} \frac{1}{(md)^s} \\ &= \zeta(s) \sum_{d \mid k} \mu\left(\frac{k}{d}\right) d^{1-s} \end{aligned} \tag{1.5.5}$$

We can also sum series of the form[②]

$$\sum_{n=1}^{\infty} \frac{c_k(n) f(n)}{n^s}$$

where $f(n)$ is a multiplicative function. For example

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{\delta \mid k, \delta \mid n} \delta \mu\left(\frac{k}{\delta}\right)$$

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① Two more proofs are given by Hardy, Ramanujan, 137-141.

② Crum (1).

$$= \sum_{\delta \mid k} \delta \mu\left(\frac{k}{\delta}\right) \sum_{m=1}^{\infty} \frac{d(m\delta)}{(m\delta)^s}$$

$$= \zeta^2(s) \sum_{\delta \mid k} \delta^{1-s} \mu\left(\frac{k}{\delta}\right) \prod_{p \mid \delta} (l + 1 - lp^{-s})$$

if $\delta = \prod p^l$. If $k = \prod p^\lambda$ the sum is

$$k^{1-s} \prod_{p \mid k} (\lambda + 1 - \lambda p^{-s}) - \sum_{p \mid k} \left(\frac{k}{p}\right)^{1-s} \{\lambda - (\lambda - 1)p^{-s}\} \cdot$$

$$\prod_{\substack{p' \mid k \\ p' \neq p}} (\lambda + 1 - \lambda p'^{-s}) + \sum_{pp' \mid k} \left(\frac{k}{pp'}\right)^{1-s} \{\lambda - (\lambda - 1)p^{-s}\} \cdot$$

$$\{\lambda - (\lambda - 1)p'^{-s}\} \prod_{\substack{p'' \mid k \\ p'' \neq p, p'}} (\lambda + 1 - \lambda p''^{-s}) - \cdots$$

$$= k^{1-s} \prod_{p \mid k} \left\{(\lambda + 1 - \lambda p^{-s}) - \frac{1}{p^{1-s}} \{\lambda - (\lambda - 1)p^{-s}\}\right\}$$

$$= k^{1-s} \prod_{p \mid k} \left\{1 - \frac{1}{p} + \lambda\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{1-s}}\right)\right\}$$

Hence

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \zeta^2(s) k^{1-s} \prod_{p \mid k} \left\{1 - \frac{1}{p} + \lambda\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{1-s}}\right)\right\} \tag{1.5.6}$$

We can also sum

$$\sum_{n=1}^{\infty} \frac{c_k(qn) f(n)}{n^s}$$

For example, in the simplest case $f(n) = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\delta \mid k, \delta \mid qn} \delta \mu\left(\frac{k}{\delta}\right)$$

For given δ, n runs through those multiples of δ/q which are integers. If δ/q in its lowest terms is δ_1/q_1, these are the numbers $\delta_1, 2\delta_2, \cdots$ Hence the sum is

$$\sum_{\delta \mid k} \delta \mu\left(\frac{k}{\delta}\right) \sum_{r=1}^{\infty} \frac{1}{(r\delta_1)^s} = \zeta(s) \sum_{\delta \mid k} \delta \mu\left(\frac{k}{\delta}\right) \delta_1^{-s}$$

Since $\delta_1 = \delta/(q, \delta)$, the result is

$$\sum_{n=1}^{\infty} \frac{c_k(qn)}{n^s} = \zeta(s) \sum_{\delta \mid k} \delta^{1-s} \mu\left(\frac{k}{\delta}\right) (q, \delta)^s \tag{1.5.7}$$

1.6. There is another class of identities involving infinite series of zeta-functions. The simplest of these is[①]

① See Landau and Walfisz (1), Estermann (1)(2).