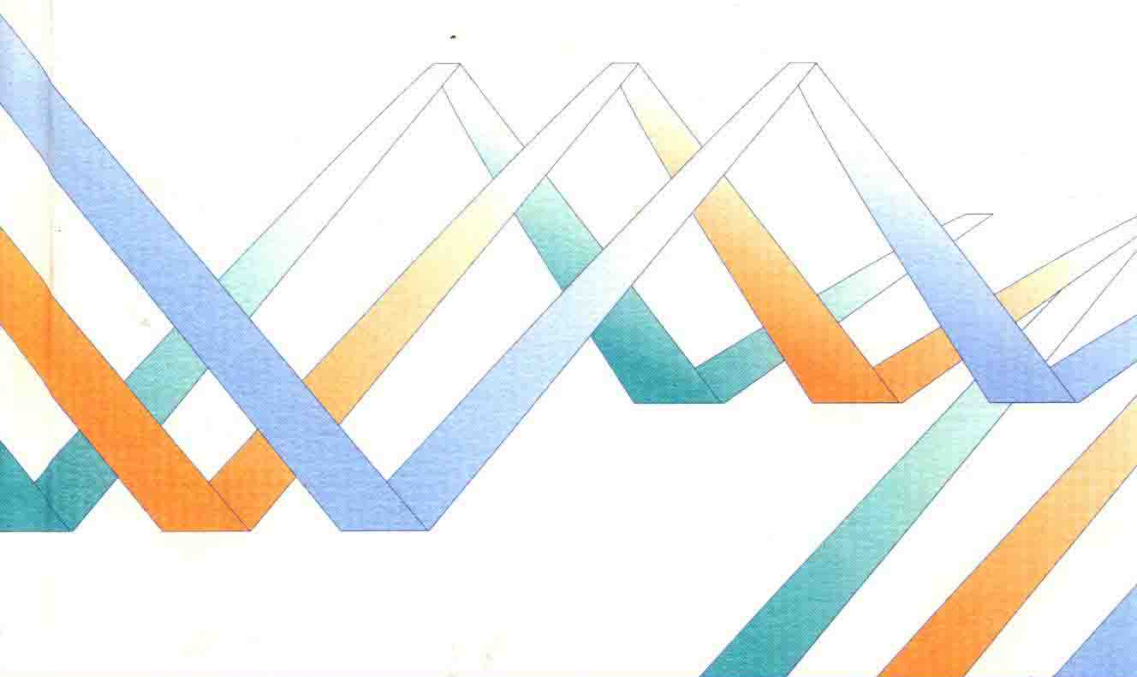


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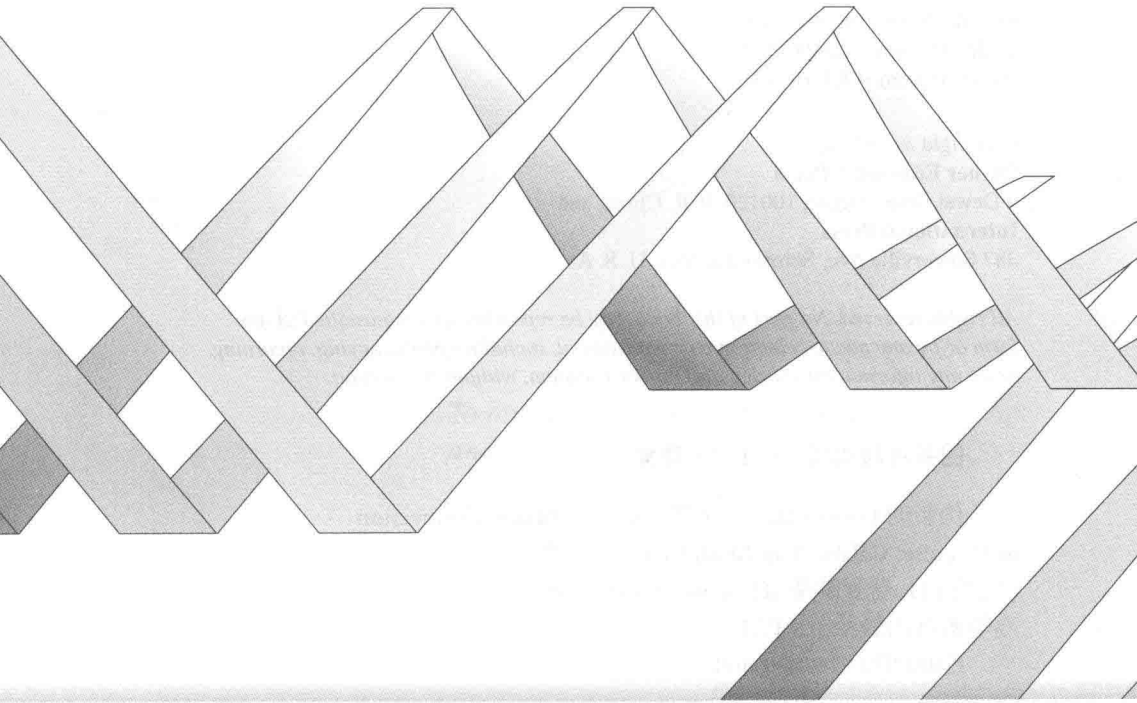
Gauss-Manin Connection in Disguise: Calabi-Yau Modular Forms

伪装的 Gauss-Manin 联络

Hossein Movasati



Higher Education Press



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Hossein Movasati

With Appendices by Khosro M. Shokri and Carlos Matheus

Author

Hossein Movasati

IMPA — Instituto de Matematica Pura e Aplicada

Estrada Dona Castorina 110

Jardim Botânico 22460-320

Rio de Janeiro — RJ Brasil

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38402 Saint-Martin d'Hères, France

Lizhen Ji

Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI, USA

Eduard J.N. Looijenga

Mathematics Department
Universiteit Utrecht
Postbus 80.010 3508 TA
Utrecht Nederland

Yat-Sun Poon

Department of Mathematics
Surge Building, 202 Surge
University of California at Riverside
Riverside, CA 92521, USA

Neil Trudinger

Centre for Mathematics
and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, ACT 0200, Australia

Jie Xiao

Department of Mathematics
Tsinghua University
Beijing 100084, China

*To my parents Rogayeh and Ali,
and my family Sara and Omid*

Preface

The guiding principle in this monograph is to develop a theory of (Calabi-Yau) modular forms parallel to the classical theory of (elliptic) modular forms. It is originated from many period manipulations of the B -model Calabi-Yau variety of mirror symmetry in Topological String Theory and the earlier works of the author in which the theory of (quasi) modular forms is introduced using a larger moduli space of elliptic curves, and the Ramanujan differential equations between Eisenstein series have been derived from the corresponding Gauss-Manin connection. We have in mind an audience with a basic knowledge of Complex Analysis, Differential Equations, Algebraic Topology and Algebraic Geometry. Although the text is purely mathematical and no background in String Theory is required, some of our computations are inspired by mirror symmetry, and so the reader who wishes to explore the motivations, must go to the original Physics literature. The text is mainly written for two primary target audiences: experts in classical modular and automorphic forms who wish to understand the q -expansions of physicists derived from Calabi-Yau threefolds, and mathematicians in enumerative Algebraic Geometry who want to understand how mirror symmetry counts rational curves in compact Calabi-Yau threefolds. Experts in modular forms are warned that they will not find so much Number Theory in the present text, as this new theory of modular forms lives its infancy, and yet many problems of complex analysis nature are open. We have still a long way to deal with more arithmetic oriented questions. For our purpose we have chosen a particular class of such q -expansions arising from the periods of a Calabi-Yau threefold called mirror quintic, and in general, periods which satisfy fourth-order differential equations. The applications of classical modular forms are huge and we are guided by the fact that this new type of modular forms might have similar applications in the near future, apart from counting rational curves and Gromov-Witten invariants. The main goal is to describe in detail many analogies and differences between classical modular forms and those treated here. The present text is a complement to the available books on the mathematical aspects of mirror symmetry such as “Mirror Symmetry and Algebraic Geometry” of D. A. Cox and S. Katz and “Mirror Symmetry” of C. Voisin. We hope that our text makes a part of mirror symmetry, which is relevant to Number Theory, more accessible to mathematicians.

Hossein Movasati
December 2016
Rio de Janeiro, RJ, Brazil

Acknowledgements

The present text is written during the years between 2010 and 2014. First of all I would like to thank Pierre Deligne for all his comments on the origin [Mov12b] of the present text. My sincere thanks go to Étienne Ghys who drew my attention to the historical aspects of Ramanujan differential equations, that is, the contribution of Darboux and Halphen which is mainly neglected in Number Theory. My sincere thanks go to Charles Doran, Stefan Reiter, Duco van Straten, Don Zagier, Conan Leung, Bong Lian and Babak Haghighat for useful conversations and their interest on the topic of the present text. I would also like to thank mathematics institutes Instituto de Matemática Pura e Aplicada (IMPA) in Rio de Janeiro (my home institute), Max-Planck Institute for Mathematics (MPIM) in Bonn, Institute for Physics and Mathematics (IPM) in Tehran, Institute for Mathematical Sciences (IMS) in Hong Kong, Center of Advanced Study in Theoretical Sciences (CASTS) in Taipei and Center of Mathematical Sciences and Applications (CMSA) at Harvard University for providing excellent research ambient during the preparation of the present text. Final versions of the present text are written when the author was spending a sabbatical year at Harvard University. Here, I would like to thank Shing-Tung Yau for the invitation and for his interest and support. My sincere thanks go to Murad Alim and Emmanuel Scheidegger for many useful conversations in mathematical aspects of Topological String Theory and mirror symmetry. I would like to thank Cumrun Vafa for his generous emails helping me to understand the mathematical content of anomaly equations. During the preparation of Appendix C we enjoyed discussions with many people. We would like to thank M. Belolipetsky for taking our attention to the works on thin groups. Thanks also go to Peter Sarnak, Fritz Beukers and Wadim Zudilin for comments on the first draft of this appendix. The second author of this appendix would like to thank CNPq-Brazil for financial support and IMPA for its lovely research ambient.

Frequently used notations

$(\mathbb{C}^n, 0)$	A small neighborhood of 0 in \mathbb{C}^n .
k, \bar{k}	A field of characteristic zero and its algebraic closure.
M^{tr}	The transpose of a matrix M . We also write $M = [M_{ij}]$, where M_{ij} is the (i, j) entry of M . The indices i and j always count the rows and columns, respectively.
V^\vee	The dual of an R -module V , where R is usually the ring \mathbb{Z} or the field k . We always write a basis of a free R -module of rank r as a $r \times 1$ matrix. For a basis δ of V and α of V^\vee we denote by
$[\delta, \alpha^{\text{tr}}] := [\alpha_j(\delta_i)]_{i,j}$	
	the corresponding $r \times r$ matrix.
d	The differential operator or a natural number, being clear in the text which one we mean.
X	A mirror quintic Calabi-Yau threefold, mirror quintic for short, or an elliptic curve defined over the field k , being clear in the text which we mean, §3.1. We will also use X as one of the Yamaguchi-Yau variables in §2.17. Another usage of X is as a fundamental system of a linear differential equation, §7.4.
$X(k)$	The set of k -rational points of X defined over the field k . In particular for $k \subset \mathbb{C}$, $X(\mathbb{C})$ is the underlying complex manifold of X . Sometimes, for simplicity we write $X = X(\mathbb{C})$, being clear in the context that X is a complex manifold.
ω, η	A differential 3-form on X . In many cases it is a holomorphic $(3, 0)$ -form.
$H_3(X, \mathbb{Z})$	The third homology of $X = X(\mathbb{C})$.
$\delta_i, i = 1, 2, 3, 4$	A symplectic basis of $H_3(X, \mathbb{Z})$.
T	Moduli of enhanced mirror quintics over \bar{k} , §2.3.
S	A moduli of mirror quintics over \bar{k} , §3.2.
$\mathcal{O}_T, \mathcal{O}_S$	The k -algebra of regular functions on T and S , $T = \text{Spec}(\mathcal{O}_T)$, etc.

$t_0, i = 0, \dots, 9$	Regular functions in T , §3.2, §3.6.
R	Vector fields on T .
Ω_T^i	The \mathcal{O}_T -module of differential i -forms on T .
$X \rightarrow T, X/T$	The universal family of enhanced mirror quintics. A single variety is denoted by X , §3.6.
$X_t, t \in T$	A fiber of $X \rightarrow T$.
$H_{\text{dR}}^*(X)$	Algebraic de Rham cohomology, §3.6.
$H_{\text{dR}}^*(X/T)$	Relative algebraic de Rham cohomology, §3.6.
$F^*H_{\text{dR}}^*(X)$	The pieces of Hodge filtration, §3.6.
$F^*H_{\text{dR}}^*(X/T)$	The pieces of Hodge filtration, §3.6.
$\alpha_i, i = 1, \dots, 4$	A basis of $H_3(X)$ or $H^3(X/T)$, §2.3.
$\langle \cdot, \cdot \rangle$	The intersection form in de Rham cohomology and singular homology, §3.4, §4.1.
$\delta^{\text{pd}} \in H_{\text{dR}}^3(X)$	The Poincaré dual of $\delta \in H_3(X, \mathbb{Z})$, §4.1.
Φ, Ψ	$\Phi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (0.1)$
G, g	An algebraic group and its element, §3.10.
$\text{Lie}(G), g$	The Lie algebra of G and its element, §3.10.
$g_i, i = 1, \dots, 6$	A parametrization of $g \in G$, §3.10.
∇	Gauss-Manin connection, §3.3, §3.8.
A	The Gauss-Manin connection matrix, §3.3, §3.8.
Y	Yukawa coupling, §2.3.
E_k	Eisenstein series of weight k , §2.5
θ	The derivation $z \frac{\partial}{\partial z}$, §2.7, or a special period matrix, §4.12.
M_0, M_1	Monodromy matrices, §2.7.
Γ	The monodromy group, §2.7
\mathbb{H}	Upper half plane or the monodromy covering, §4.6.
τ	The canonical coordinate of the upper half plane or a special period matrix, §4.2.
$\tau_i, i = 0, 1, 2, 3$	Meromorphic functions on \mathbb{H} , §2.10, §10.2.
$\theta_i, i = 0, 1, 2, 3$	Meromorphic functions on \mathbb{H} , §4.12, §10.2.
$F_g^{\text{alg}}, F_g^{\text{non}}, F_g^{\text{hol}}$	Genus g topological string partition functions, §2.13.
L	A fourth-order linear differential equations, §2.15.
ψ_0, ψ_1	Holomorphic and logarithmic solutions of $L = 0$, §2.7, §7.1.
x_{ij}	Particular solutions of $L = 0$ and their derivatives, §2.7, §7.4. We denote by $X := [x_{ij}]$ the corresponding matrix.
y_{ij}	General solutions of $L = 0$ and their derivatives, §4.15.
A	A linear system attached to $L = 0$, §7.3.
$a_i(z)$	Coefficients of L , §2.15.
u_i	Expressions in terms of the solutions of L , §2.15.
X, U, V_1, V_2, V_3	Yamaguchi-Yau variables, §2.17, §6.1, §8.4.

P	Period map or period matrix, §4.1.
Π	Generalized period domain, §4.4.
F, G, E	Intersection matrices, §7.4.
χ	The Euler number of the underlying Calabi-Yau threefolds of $L = 0$, §7.3.
$h^{2,1}$	The $(2, 1)$ Hodge number of the underlying Calabi-Yau threefolds of $L = 0$. The text mainly deals with $h^{2,1} = 1$.

Online supplemental items

Many arguments and proofs of the present text rely on heavy computer computations. For this purpose, we have used Singular, [GPS01], a computer programming language for polynomial computations. Throughout the text I have used notations in the form [Supp Item x], where $x=1, 2, 3$, etc. Each of these notations refers to a supplemental item, such as computer data or code, etc., that can be accessed online from my web page:

<http://w3.impa.br/~hossein/singular/GMCD-MQCY3-SuppItems.html>

The web page serves as a hot-linked index to all of this book's online supplemental items. These supplemental items are mainly for two purposes. First, they are mainly for the reader who does not want to program by her or himself and wants to check the statements using our computer codes. Second, we only present a small amount of computer data in the text, and the reader can use the supplemental item in order to access to more data, for instance more coefficients of a q -expansion of a series. We have also written a library in Singular [Supp Item 1] which has been useful for our computations.

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