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Surveys of Modern Mathematics

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Calabi-Yau Modular Forms

伪装的 Gauss-Manin 联络

Hossein Movasati

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Surveys of Modern Mathematics

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Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

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Series Editors

To my parents Rogayeh and Ali,
and my family Sara and Omid

Preface

The guiding principle in this monograph is to develop a theory of (Calabi-Yau) modular forms parallel to the classical theory of (elliptic) modular forms. It is originated from many period manipulations of the B-model Calabi-Yau variety of mirror symmetry in Topological String Theory and the earlier works of the author in which the theory of (quasi) modular forms is introduced using a larger moduli space of elliptic curves, and the Ramanujan differential equations between Eisenstein series have been derived from the corresponding Gauss-Manin connection. We have in mind an audience with a basic knowledge of Complex Analysis, Differential Equations, Algebraic Topology and Algebraic Geometry. Although the text is purely mathematical and no background in String Theory is required, some of our computations are inspired by mirror symmetry, and so the reader who wishes to explore the motivations, must go to the original Physics literature. The text is mainly written for two primary target audiences: experts in classical modular and automorphic forms who wish to understand the q-expansions of physicists derived from Calabi-Yau threefolds, and mathematicians in enumerative Algebraic Geometry who want to understand how mirror symmetry counts rational curves in compact Calabi-Yau threefolds. Experts in modular forms are warned that they will not find so much Number Theory in the present text, as this new theory of modular forms lives its infancy, and yet many problems of complex analysis nature are open. We have still a long way to deal with more arithmetic oriented questions. For our purpose we have chosen a particular class of such q-expansions arising from the periods of a Calabi-Yau threefold called mirror quintic, and in general, periods which satisfy fourth-order differential equations. The applications of classical modular forms are huge and we are guided by the fact that this new type of modular forms might have similar applications in the near future, apart from counting rational curves and Gromov-Witten invariants. The main goal is to describe in detail many analogies and differences between classical modular forms and those treated here. The present text is a complement to the available books on the mathematical aspects of mirror symmetry such as "Mirror Symmetry and Algebraic Geometry" of D. A. Cox and S. Katz and "Mirror Symmetry" of C. Voisin. We hope that our text makes a part of mirror symmetry, which is relevant to Number Theory, more accessible to mathematicians.

Hossein Movasati
December 2016
Rio de Janeiro, RJ, Brazil

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Frequently used notations

$(\mathbb{C}^n, 0)$	A small neighborhood of 0 in $\mathbb{C}^n$.
$\mathsf{k}, \bar{\mathsf{k}}$	A field of characteristic zero and its algebraic closure.
M^{tr}	The transpose of a matrix M. We also write $M = [M_{ij}]$, where M_{ij} is the (i, j) entry of M. The indices i and j always count the rows and columns, respectively.
$V^\vee$	The dual of an R-module V, where R is usually the ring $\mathbb{Z}$ or the field k. We always write a basis of a free R-module of rank r as a $r \times 1$ matrix. For a basis δ of V and α of $V^\vee$ we denote by $$[\delta, \alpha^{\mathrm{tr}}] := [\alpha_j(\delta_i)]_{i,j}$$ the corresponding $r \times r$ matrix.
d	The differential operator or a natural number, being clear in the text which one we mean.
X	A mirror quintic Calabi-Yau threefold, mirror quintic for short, or an elliptic curve defined over the field k, being clear in the text which we mean, §3.1. We will also use X as one of the Yamaguchi-Yau variables in §2.17. Another usage of X is as a fundamental system of a linear differential equation, §7.4.
$X(\mathsf{k})$	The set of k-rational points of X defined over the field k. In particular for $\mathsf{k} \subset \mathbb{C}$, $X(\mathbb{C})$ is the underlying complex manifold of X. Sometimes, for simplicity we write $X = X(\mathbb{C})$, being clear in the context that X is a complex manifold.
ω, η	A differential 3-form on X. In many cases it is a holomorphic $(3,0)$-form.
$H_3(X, \mathbb{Z})$	The third homology of $X = X(\mathbb{C})$.
$\delta_i,\ i = 1,2,3,4$	A symplectic basis of $H_3(X, \mathbb{Z})$.
T	Moduli of enhanced mirror quintics over $\bar{\mathsf{k}}$, §2.3.
S	A moduli of mirror quintics over $\bar{\mathsf{k}}$, §3.2.
$\mathscr{O}_\mathsf{T}, \mathscr{O}_\mathsf{S}$	The k-algebra of regular functions on T and S, $\mathsf{T} = \mathrm{Spec}(\mathscr{O}_\mathsf{T})$, etc.

$t_0,\ i=0,\dots,9$	Regular functions in T, §3.2, §3.6.
R	Vector fields on T.
Ω^i_{T}	The $\mathscr{O}_{\mathsf{T}}$-module of differential i-forms on T.
$\mathsf{X}\to\mathsf{T},\ \mathsf{X}/\mathsf{T}$	The universal family of enhanced mirror quintics. A single variety is denoted by X, §3.6.
$\mathsf{X}_t,\ t\in\mathsf{T}$	A fiber of $\mathsf{X}\to\mathsf{T}$.
$H^*_{\mathrm{dR}}(X)$	Algebraic de Rham cohomology, §3.6.
$H^*_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$	Relative algebraic de Rham cohomology, §3.6.
$F^*H^*_{\mathrm{dR}}(X)$	The pieces of Hodge filtration, §3.6.
$F^*H^*_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$	The pieces of Hodge filtration, §3.6.
$\alpha_i,\ i=1,\dots,4$	A basis of $H_3(X)$ or $H^3(\mathsf{X}/\mathsf{T})$, §2.3.
$\langle\cdot,\cdot\rangle$	The intersection form in de Rham cohomology and singular homology, §3.4, §4.1.
$\delta^{\mathsf{pd}}\in H^3_{\mathrm{dR}}(X)$	The Poincaré dual of $\delta\in H_3(X,\mathbb{Z})$, §4.1.
Φ,Ψ	

$$\Phi:=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\quad \Psi:=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix} \tag{0.1}$$

G,g	An algebraic group and its element, §3.10.
$\mathrm{Lie}(\mathsf{G}),\ \mathfrak{g}$	The Lie algebra of G and its element, §3.10.
$\mathsf{g}_i,\ i=1,\dots,6$	A parametrization of $\mathsf{g}\in\mathsf{G}$, §3.10.
∇	Gauss-Manin connection, §3.3, §3.8.
A	The Gauss-Manin connection matrix, §3.3, §3.8.
Y	Yukawa coupling, §2.3.
E_k	Eisenstein series of weight k, §2.5
θ	The derivation $z\frac{\partial}{\partial z}$, §2.7, or a special period matrix, §4.12.
M_0,M_1	Monodromy matrices, §2.7.
Γ	The monodromy group, §2.7
$\mathbb{H}$	Upper half plane or the monodromy covering, §4.6.
τ	The canonical coordinate of the upper half plane or a special period matrix, §4.2.
$\tau_i,\ i=0,1,2,3$	Meromorphic functions on $\mathbb{H}$, §2.10, §10.2.
$\theta_i,\ i=0,1,2,3$	Meromorphic functions on $\mathbb{H}$, §4.12, §10.2.
$\mathsf{F}^{\mathrm{alg}}_g,\mathsf{F}^{\mathrm{non}}_g,\mathsf{F}^{\mathrm{hol}}_g$	Genus g topological string partition functions, §2.13.
L	A fourth-order linear differential equations, §2.15.
ψ_0,ψ_1	Holomorphic and logarithmic solutions of $L=0$, §2.7, §7.1.
x_{ij}	Particular solutions of $L=0$ and their derivatives, §2.7, §7.4. We denote by $\mathrm{X}:=[x_{ij}]$ the corresponding matrix.
y_{ij}	General solutions of $L=0$ and their derivatives, §4.15.
A	A linear system attached to $L=0$, §7.3.
$a_i(z)$	Coefficients of L, §2.15.
u_i	Expressions in terms of the solutions of L, §2.15.
X,U,V_1,V_2,V_3	Yamaguchi-Yau variables, §2.17, §6.1, §8.4.

P	Period map or period matrix, §4.1.
Π	Generalized period domain, §4.4.
F, G, E	Intersection matrices, §7.4.
χ	The Euler number of the underlying Calabi-Yau threefolds of $L=0$, §7.3.
h^{21}	The $(2,1)$ Hodge number of the underlying Calabi-Yau threefolds of $L=0$. The text mainly deals with $h^{21}=1$.

Online supplemental items

Many arguments and proofs of the present text rely on heavy computer computations. For this purpose, we have used Singular, [GPS01], a computer programming language for polynomial computations. Throughout the text I have used notations in the form [Supp Item x], where x=1, 2, 3, etc. Each of these notations refers to a supplemental item, such as computer data or code, etc., that can be accessed online from my web page:

http://w3.impa.br/~hossein/singular/GMCD-MQCY3-SuppItems.html

The web page serves as a hot-linked index to all of this book's online supplemental items. These supplemental items are mainly for two purposes. First, they are mainly for the reader who does not want to program by her or himself and wants to check the statements using our computer codes. Second, we only present a small amount of computer data in the text, and the reader can use the supplemental item in order to access to more data, for instance more coefficients of a q-expansion of a series. We have also written a library in Singular [Supp Item 1] which has been useful for our computations.

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