Günter Harder

Lectures on Algebraic Geometry I

Sheaves, Cohomology of Sheaves, and Applications to Riemann Surfaces
2nd Edition

代数几何讲义 第1卷

第2版

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著　　者：Günter Harder
责任编辑：刘　慧　高　蓉
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Prof. Dr. Günter Harder
Max-Planck-Institute for Mathematics
Vivatsgasse 7
53111 Bonn
Germany

harder@mpim-bonn.mpg.de

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Preface

I want to begin with a defense or apology for the title of this book. It is the first part of a two volume book. The two volumes together are meant to serve as an introduction into modern algebraic geometry. But about two thirds of this first volume concern homological algebra, cohomology of groups, cohomology of sheaves and algebraic topology. These chapters 1 to 4 are more an introduction into algebraic topology and homological algebra than an introduction into algebraic geometry. Only in the last Chapter 5 we will see some algebraic geometry. In this last chapter we apply the results of the previous sections to the theory of compact Riemann surfaces. Even this section does not look like an introduction into modern algebraic geometry, large parts of the material covered looks more like 19'th century mathematics. But historically the theory of Riemann surfaces is one of the roots of algebraic geometry.
We will prove the Riemann-Roch theorem and we will discuss the structure of the divisor class group. These to themes are ubiquitous in algebraic geometry. Finally I want to say that the theory of Riemann surfaces is also in these days a very active area, it plays a fundamental role in recent developments. The moduli space of Riemann surfaces attracts the attention of topologists, number theorists and of mathematical physicists. To me this seems to be enough justification to begin an introduction to algebraic geometry by discussing Riemann surfaces at the beginning.

Only in the second volume we will lay the foundations of modern algebraic geometry. We introduce the notion of schemes, I discuss the category of schemes, morphisms and so on. But as we proceed the concepts of sheaves, cohomology of sheaves and homological algebra, which we developed in this first volume, will play a predominant role. We will resume the discussion of the Riemann-Roch theorem and discuss the Picard group or jacobians of curves.

A few more words of defense. These books grew out of some series of lectures, which I gave at the university of Bonn. The first lectures I gave were lectures on cohomology of arithmetic groups and it was my original plan to write a book on the cohomology of arithmetic groups. I still have the intention to do so. Actually there exists a first version of such a book. It consists of a series of notes taken from a series of lectures I gave on this subject. Arithmetic groups Γ are groups of the form $\Gamma = \mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$ or the symplectic group $\Gamma = \mathrm{Sp}_n(\mathbb{Z}) \subset \mathrm{Sp}_n(\mathbb{R})$(See 5.2.24). These groups act on the symmetric spaces $X = G(\mathbb{R})/K_\infty$ and the quotient spaces $\Gamma\backslash X$. The representations of the algebraic group G define sheaves $\widetilde{M}$ on this space and the cohomology groups $H^\bullet(\Gamma\backslash X,\widetilde{M})$ will be investigated in this third volume. Again the results in the first four chapters of the first volume will be indispensable.

But in this third volume we will also need some background in algebraic geometry. In some cases the quotient spaces $\Gamma\backslash X$ carry a complex structures, these are the Shimura varieties. Then it is important to know, that these quotients are actually quasi projective algebraic varieties and that they are defined over a much smaller field, namely a number field. To understand, why this is so, we interpret this spaces as parameter spaces of certain algebraic objects, i.e. they turn out to be " moduli spaces", especially the moduli spaces of abelian varieties. This last subject is already briefly touched in this first volume and will be resumed in the second and third volume.

Perhaps this is the right moment to confess that I consider myself as a number theorist. Number theory is a broad field and for the kind of questions, I am interested in, the methods and concepts algebraic geometry, cohomology of arithmetic groups, the theory of automorphic forms are essential. Therefore it is my hope that these three volumes together can serve as an introduction into an interesting branch of mathematics.

This book is addressed to students who have some basic knowledge in analysis, algebra and basic set theoretic topology. So a student at a German university can read it after the second year at the university.

I want to thank my former student Dr. J. Schlippe, who went through this manuscript many times and found many misprint and suggested many improvements. I also thank J. Putzka who "translated" the original Plain-Tex file into Latex and made it consistent with the demands of the publisher. But he also made many substantial suggestions concerning the exposition and corrected some errors.

Günter Harder Bonn, December 2007

Preface to the second edition

In the meantime the second volume of this book appeared and the publisher decided to prepare a second edition of this first volume.

For this new edition I corrected a few misprints and modified the exposition at some places. I also added a short section on moduli of elliptic curves with N-level structures. Here I followed closely the presentation of this subject in the Diploma thesis of my former student Christine Heinen.

This new paragraph anticipates some of the techniques of volume II. I originally planned to include it into the second Volume. Since I already had a section on moduli of elliptic curves with a differential and since the second volume became too long I abandoned this plan. Therefore, I was quite happy when I got the opportunity to include this section into the second edition of the first volume. It also helps a little bit to keep the balance between the two volumes. This moduli space and some generalizations of it will play a role in my book on "Cohomology of arithmetic groups".

Günter Harder Bonn, June 2011

Introduction

This first volume starts starts with a very informal introduction into category theory. It continues with an introduction into homological algebra. In view of the content of the third volume Chapter 2 is an introduction into homological algebra based on the example of cohomology of groups.

Chapter 3 introduces into the theory of sheaves. The role of sheaves is twofold: They allow us to formulate the concepts of manifolds as locally ringed spaces ($\mathcal{C}^\infty$-manifolds, complex manifolds, algebraic manifolds...); this is discussed in section 3.2. The concept of locally ringed space will be indispensable when we introduce the concept of schemes in the second volume.

The second role is played by the cohomology of sheaves which is covered in Chapter 4. My original notes gave only a very informal introduction into sheaf cohomology, but after a while I felt the desire to give a rather self contained account. So it happened that the introduction into sheaf cohomology became rather complete up to a certain level. I included spectral sequences, the cup product and the Poincaré duality of local systems on manifolds. I also discuss intersection products and the Lefschetz fixed point formula for some special cases. So it happened that Chapter 4 became very long and it has several subsections. Up to Chapter 4.7 the book may serve as an introduction into algebraic topology but with a strong focus on applications to algebraic geometry and to the cohomology of arithmetic groups. The discussion of singular homology is rather short.

In the final sections of Chapter 4 I discuss the analytic methods in the study of cohomology of manifolds. I discuss the de Rham isomorphism, which gives a tool to understand the cohomology of local systems. In analogy to that the Dolbeault isomorphism gives us an instrument to investigate the cohomology of holomorphic bundles on complex manifolds. Finally I explain the basic ideas of Hodge theory. Only in the section on Hodge theory I need to refer to some analytical results which are not proved in this book.

The last chapter 5 we apply these results and concepts to the theory of compact Riemann surfaces. In the first section of Chapter 5 we prove the theorem of Riemann-Roch. We want to make it clear that the hardest part in the proof of the theorem of Riemann-Roch is the finite dimensionality of some cohomology groups and this proof requires some difficult analysis. We also give some indications how these analytic results can be proved in our special case. From the theorem of Riemann-Roch it follows, that Riemann surfaces may be viewed as purely algebraic objects, we prove that they are smooth projective algebraic curves. At this point we see some concepts of commutative algebra entering the stage. They will be discussed in more detail in volume II. We discuss Abel's theorem which explains the structure of the divisor class group. It turns out that the group of divisor classes of degree zero is a complex torus with a principal polarization (Riemann Period relations), this says that it is an abelian variety over $\mathbb{C}$.

In the second section of Chapter 5 we discuss the meaning of this fact. We examine line bundles on these Jacobians and more general line bundles on abelian varieties. Especially we describe the spaces of sections of line bundles in terms of spaces of theta-series. We also explain in a very informal way the relationship to the moduli spaces of principally polarized abelian varieties. I also have a section on the theory of Jacobi-Theta-functions. This is the one dimensional case. It illustrates the connections to very old and classical mathematics. But in the back of my mind I see this also as a preparation for the book on cohomology of arithmetic groups. To say this differently, we see the connections between the moduli spaces of abelian varieties and the theory of modular forms.

This last chapter goes beyond homological algebra and algebraic topology. But it shows the enormous usefulness of these concepts. Chapter 5 can also be seen as a preparation for the second volume, which is an introduction into algebraic geometry. In the second half of Chapter 5 we discuss the structure of Jacobians, their Neron-Severi groups and the structure of endomorphism rings. These arguments and methods will appear again in the second volume, when we discuss the Jacobians of curves over arbitrary fields. In the last section of this first volume we give some outlook on celebrated results, which will also not be proved in the second volume, but for whose proof we provide some preparation.

Contents

1 Categories, Products, Projective and Inductive Limits

1.1 The Notion of a Category and Examples

I want to give a very informal introduction to the theory of categories. The main problem for a beginner is to get some acquaintance with the language and to get used to the abstractness of the subject. As a general reference I give the book [McL].

Definition 1.1.1. *A* **category** $\mathcal{C}$ *is*

(i) *a collection of objects* $\mathrm{Ob}(\mathcal{C})$.

We do not insist that this collection is a set. For me this means that we do not have the notion of equality of two objects. If we write $N \in \mathrm{Ob}(\mathcal{C})$ *then we mean that* N *is an object in the category* $\mathcal{C}$.

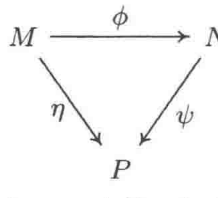
(ii) *To any two objects* $N,M \in \mathrm{Ob}(\mathcal{C})$ *there is attached a set* $\mathrm{Hom}_{\mathcal{C}}(N,M)$ *which is called the set of* **morphisms** *between these two objects.*

Usually we denote a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(N,M)$ *by an arrow* $\phi : N \longrightarrow M$.

(iii) *For any three objects* N,M,P *we have the* **composition** *of morphisms*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(N,M) \times \mathrm{Hom}_{\mathcal{C}}(M,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(N,P) \\ (\phi,\psi) & \longmapsto & \psi \circ \phi. \end{array}$$

If a morphism η *is a composition of* ϕ *and* ψ *then we denote this by a* **commutative diagram** *(or commutative triangle)*



We require that this composition is associative in the obvious sense (if we have four objects...). The reader should verify that this associativity can be formulated in terms of a tetrahedron all of whose four sides are commutative triangles. Here we use that the morphisms between objects form a set. In a set we know what equality between elements means.

(iv) *For any object* $N \in \mathrm{Ob}(\mathcal{C})$ *we have a distinguished element* $\mathrm{Id}_N \in \mathrm{Hom}_{\mathcal{C}}(N,N)$, *which is an identity on both sides under the composition.*

Everybody has seen the following categories

Example 1. *The category* **Ens** *of all sets where the arrows are arbitrary maps.*

Example 2. *The category* $\mathbf{Vect}_k$ *of vector spaces over a given field* k *where the sets of morphisms are the* k*-linear maps.*

Example 3. *The category* $\mathbf{Mod}_A$ *of modules over a ring* A *where the morphisms are* A*-linear maps. We also have the category of abelian groups* **Ab**, *the category* **Groups** *of all groups where the morphisms are the homomorphisms of groups.*

Example 4. *The category* **Top** *of topological spaces where the morphisms are the continuous maps.*

I said in the beginning that we do not have the notion of equality of two objects M,N in a category. But we we can say that two objects $N,M \in \mathrm{Ob}(\mathcal{C})$ are **isomorphic.** This means that we can find two arrows $\phi : N \longrightarrow M$ and $\psi : M \longrightarrow N$ such that $\mathrm{Id}_N = \psi \circ \phi, \mathrm{Id}_M = \phi \circ \psi$. But in general it may be possible to find many such isomorphisms between the objects and hence we have many choices to identify them. Then it is better to refrain from considering them as equal.
For instance we can consider the category of finite dimensional vector spaces over a field k. Of course two such vector spaces are isomorphic if they have the same dimension. Since we may have many of these isomorphisms, we do not know how to identify them and therefore the notion of equality does not make sense.
But if we consider the category of framed finite dimensional $k-$vector spaces, i.e. vector spaces V equipped with a basis which is indexed by the numbers $1,2,\dots,n = \dim(V)$. Now morphisms which are linear maps which send basis elements to basis elements and which respect the ordering. Then the situation is different. We can say the objects form a set: If two such objects are isomorphic then the isomorphism is unique.

It is important to accept the following fact: The axioms give us a lot of flexibility, at no point we require that the elements in $\mathrm{Hom}_{\mathcal{C}}(N,M)$ are actual maps between sets (with some additional structure). Insofar all the above examples are somewhat misleading.
A simple example of a situation where the arrows are not maps is the following one:

Example 5. *We may start from a an ordered set* $\mathcal{I} = (I, \leq)$ *and we consider its elements as the objects of a category. For any pair* $i,j \in I$ *we say that* $\mathrm{Hom}_{\mathcal{I}}(i,j)$ *consists of one single element* $\phi_{i,j}$ *if* $i \leq j$ *and is empty otherwise. The composition is the obvious one obtained from the transitivity of the order relation.*

The reader may say that this is not a good example, because the $\phi_{i,j}$ can be considered as maps between the two sets $\{i\},\{j\}$ but that is the wrong point of view. To make this clear we can also construct a slightly different category $\mathcal{J}$ from our ordered set. We assume that the order relation satisfies $i \leq j$ and $j \leq i$ implies $i = j$ and hence we can define $i < j$ by $i \leq j$ and $i \neq j$. Then we may define the sets of morphisms as:

$\mathrm{Hom}_{\mathcal{J}}(i,j)$ are finite sequences $\{i_0,i_1,\dots,i_n\}$ with $i_\nu < i_{\nu+1}$ and $i = i_0, j = i_n$.

These sequences form a set. We leave it to the reader to verify that we have a composition and an identity. Now we may have many arrows between two objects $\{i\},\{j\}$ which are sets consisting of one element.