Günter Harder

Lectures on Algebraic Geometry II

Basic Concepts, Coherent Cohomology, Curves and their Jacobians

代数几何讲义

第2卷

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Lectures on Algebraic Geometry Ⅱ

Basic Concepts，Coherent Cohomology，Curves and their Jacobians

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Günter Harder

Lectures on Algebraic Geometry II

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Preface

This is now, at last, the second volume of my "Lectures on Algebraic Geometry". When working on this second volume, I always had a saying by Peter Gabriel on my mind:

"Der Weg zur Hölle ist mit zweiten Bänden gepflastert!"

(The path to hell is paved with (never written?) second volumes.) Very often I felt like Sisyphos in Homer's Oddyssea. Sisyphos tries to push a rock over the ridge and just before he reaches top the rock rolls down again. Only at this very moment, when I am writing this preface, I am gaining some confidence that this second volume finally may come to life.

It is still valid what I said in the preface to the first volume, I plan to write a book on **Cohomology of arithmetic groups**. Actually there exists a very preliminary version of this "Volume III" on my home page at the Bonn university. The present book is also meant to provide background for "VolumeIII".
"Volume III" will be different in nature, we do not give an introduction into a field which is well established and already treated in other text books. It will rather be a description of a research area which is still developing, it will contain some new results, and it will put old results into a new perspective. I will formulate open questions and formulate problems, which are important on one hand but which are also tractable.
The first group of fundamental results in this book is proved in Chapter 8 when I discuss the finiteness results for the cohomology of coherent sheaves and the semi-continuity theorems. Here I use the theorems on sheaf cohomology which are proved in the first volume. I put a lot of emphasis on the relevance of the semi-continuity theorems for the construction of moduli spaces.

Moduli spaces are a central theme in this book. We discuss the moduli space of elliptic curves which are equipped in a nowhere vanishing differential form in Chapter 9. This moduli space and its generalization to moduli of abelian varieties will play a prominent role in "VolumeIII". On the other hand the representability of the modified Picard functor for curves (Chap. 10) is one of the main (and most difficult) results in this book.
At several places I give informal outlooks into further developments. In the last part of Chapter 9 I discuss the general version of the Grothendieck-Riemann-Roch theorem. Here I have to ask the reader to accept some concepts and results, for instance the existence of the Chow-ring, the theory of Chern classes and finally the Grothendieck-Riemann-Roch theorem itself.
The final goal of this book is to bring the reader to the foothills of the mountain range of étale cohomology. I give the definition of étale cohomology groups and "compute" these cohomology groups for curves. These first basic results on étale cohomology depend on the results proved in 10.2 and 10.3. Once we have some acquaintance with étale cohomology we can look to the giant peaks in the distance, for example the Weil conjectures and the modularity of elliptic curves. But we also see some peaks that so far nobody climbed, so for instance the Hodge and the Tate conjectures. We can define the L-functions attached to the cohomology of smooth projective algebraic varieties or even motives over number

fields. Then can formulate certain aspects of the Langlands program, these things will be discussed in "Volume III". There are excellent books which guide the reader into étale cohomology . (See [Del], [Mi], [F-K] [K-W].)
Again I want to thank my former student Dr. J. Schlippe, who went through this manuscript many times and found many misprint and suggested many improvements. I also thank J. Putzka who "translated" the original Plain-Tex file into Latex and made it consistent with the demands of the publisher. But he also made many substantial suggestions concerning the exposition and corrected some errors.

Günter Harder Bonn, February 2011

Introduction

This second volume starts where the first volume ends. In the first volume we did a lot of topology and also some analysis, in the last chapter we introduced compact Riemann surfaces. These are by definition compact complex manifolds of dimension one. But finally it turned out that they can be understood as purely algebraic objects; this is discussed in Vol. I, 5.1.7. In 5.1.14 we attach a locally ringed space to such a surface, and this locally ringed space is a scheme. This process of algebraization of analytic objects is continued in Vol. I, 5.3.

Hence we develop the language schemes in the first chapter of the second volume, and consequently this is Chapter 6 of the series. We discuss the basic abstract notions in the theory of schemes. Here the exposition has a higher level of abstractness and generality. In this chapter we also discuss the very abstract notions of descend. These notions play an important role in the last chapter. The reader may skip this part in first reading.

Chapter 7 is an introduction to commutative algebra and its implications in geometry. Here we are not very systematic and do not discuss all aspects in full generality. We only discuss very basic notions, we prove some of the easier theorems, and for the more difficult theorems we refer to the literature. As a byproduct the reader gets an introduction to algebraic number theory. We prove some of the fundamental theorems in algebraic number theory and formulate Dirichlet's theorem on units, the finiteness of the class number, and the unramified case of Artin's reciprocity law.

Chapter 8 is an introduction into projective algebraic geometry. After explaining the basic notions, we treat the fundamental finiteness theorems for the cohomology of coherent sheaves. After that we discuss the semi-continuity theorems, which fundamental in the construction of moduli spaces.

In the first part of Chapter 9 we consider projective curves, these are smooth projective varieties of dimension one. The first theme is the theorem of Riemann-Roch, here we emphasize that the theorem of Riemann-Roch, as it is usually stated, and Serre-duality should be considered as a unity. In my view these two theorems together should be called the Riemann-Roch theorem for curves. Our approach is different from the usual one, for our treatment is very close to the approach in the paper of Dedekind-Weber [De-We].

We then proceed and discuss some applications of the Riemann-Roch theorem. One of these applications concerns moduli problems. We show how the results on semi-continuity provide a tool to construct moduli spaces (elliptic curves together with a non vanishing differential, thm. 9.6.2). But after that we make some efforts to discuss the subtleties behind the notion of moduli spaces if the objects, which we want to classify, have automorphisms. This leads to the distinction between fine and coarse moduli spaces. The discussion also makes it clear that we can not hope for a moduli space of elliptic curves (or of curves of genus g). It is possible to define a more general class of objects, these are the so called stacks. It has been proved by Deligne and Mumford that the moduli stack of curves of genus g exists.

Finally we discuss the general Riemann-Roch theorem of Grothendieck. Here we can not prove everything, we have to accept the existence of the Chow ring, the theory of Chern classes and the isomorphism between two different definitions of K-groups. We formulate the general Grothendieck- Riemann-Roch theorem (GRR).

We also discuss and prove a special of GRR for products of curves over fields. Here we hope that the reader gets a glimpse of the proof of the general GRR. We use this version of GRR to prove the Hodge-index theorem for this special case.
In the last section of this Chapter we discuss curves over finite fields. In the beginning this looks rather innocent, but in my view it is a first culmination point in this book. We explain the relationship between the Riemann-Roch theorem and the Zeta-function of the curve. If we take the analogy between number fields and function fields into account, then the Zeta-function can be defined in terms of the function field. If we we look closer into this analogy-here we recommend strongly to read Neukirch's exposition in [Neu], Chapter VII. Then we see that the Riemann-Roch theorem (in the above sense) is essentially the Poisson summation formula and that this formula is the the basic reason for the functional equation of the Zeta-function.
But we go one step further and we give a proof of the analogue of the Riemann hypothesis, which in the realm of algebraic geometry is called the "Weil conjecture". Here we reproduce the arguments of Mattuck-Tate in [Ma-Ta] and of Grothendieck in [Gr-RH] and we show how the Riemann hypothesis follows from the Hodge index theorem applied to the product of the curve by itself.

In the last Chapter we discuss the the Picard functor on curves, in other words we investigate line bundles, or better the totality of line bundles on a given curve C/k. The first theorem is the the representability of some slightly modified Picard functors. This is a hard piece of work.
We prove that $\mathrm{Pic}^0_{C/k}$ is an abelian variety defined over k, this means it is a connected projective variety together with the structure of a commutative group scheme. It is called the Jacobian of the curve.
Starting from there we develop the theory of abelian varieties, and we study the Picard scheme of abelian varieties, we investigate their endomorphism rings and the ℓ-adic representation. This exposition overlaps with the book [Mu1], but we start from the Jacobians as prototypes of abelian varieties, whereas Mumford stubbornly avoids to speak of Jacobians.
Finally we give an outlook to the étale cohomology of schemes. We explain the concepts and formulate some of the basic theorems. Especially we formulate Deligne's theorem, i.e. we give the formulation of the Weil conjectures for smooth projective varieties. We prove this theorem (in a certain sense) for abelian varieties and for curves.

We conclude by discussing a degenerating family of elliptic curves. The purpose of this example is twofold. Firstly: Understanding such degenerations is important for the compactification of moduli spaces (stacks) of curves or abelian varieties. We describe in this special case how the theory of Θ -functions can be used to analyze elliptic curves in the neighborhood of their locus of degeneration, and write down explicit equations. This gives us a tool to compactify the moduli space. For the general case of abelian varieties we refer to [Fa-Ch].
Secondly we use this example to illustrate the final step in Deligne's proof of the Weil conjecture. This gives me the opportunity to finish this book with an exceptionally beautiful proof.

Edited by Klas Diederich

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Contents

6 Basic Concepts of the Theory of Schemes

6.1 Affine Schemes

We consider commutative rings $A,B,\ldots$ with identity $(1_A,1_B,\ldots)$, homomorphisms $\phi : A \to B$ are always assumed to send the identity of A into the identity of B. We always assume that the identity in a ring is different from zero. A ring A is called **integral** if it does not have zero divisors.

For any such ring A we have the group of invertible elements (units):

Definition 6.1.1. *The* **group of invertible elements (units)** *of a commutative ring with identity is defined by $A^\times = \{a \in A \mid \exists\, a' \in A$ such that $aa' = 1_A\}$. An Element in $A^\times$ is called* **unit**.

Definition 6.1.2. *A* **proper** *ideal $\mathfrak{a} \subset A$ is an ideal with $1_A \notin \mathfrak{a}$, prime ideals are always proper.*

For any ring and any $f \in A$ we use the standard notation (f) for the principal ideal Af. If we have a homomorphism $\phi : A \to B$, then we will say that B is an **A-algebra** .

6.1.1 Localization

If we have a subset $S \subset A$, which is closed under multiplication and contains the identity $1_A \in S$, we can define a quotient ring A_S and a map $\phi_S : A \longrightarrow A_S$ such that the elements of S become invertible.
To do this we consider pairs $(a,s) \in A \times S$ and introduce an equivalence relation

$$(a,s) \sim (a',s') \iff \exists\, s'' \in S \text{ such that } (as' - a's) \cdot s'' = 0. \tag{6.1}$$

We consider the quotient A_S of $A \times S$ by this relation, let $\pi_S : A \times S \longrightarrow A_S$ be the projection to this quotient. We define a ring structure on A_S by

$$\begin{aligned} \pi_S((a,s)) + \pi_S((a',s')) &= \pi_S((as' + a's,ss')) \\ \pi_S((a,s)) \cdot \pi_S((a',s')) &= \pi_S((aa',ss')). \end{aligned} \tag{6.2}$$

We have a homomorphism of rings

$$\begin{aligned} \phi_S : A &\longrightarrow A_S \\ a &\longmapsto \pi_S((a,1)). \end{aligned}$$

We will write the elements of A_S simply as

$$\pi_S((a,s)) = \frac{a}{s} = as^{-1}.$$