SMM 12
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Shiing-Shen Chern

Higher Education Press

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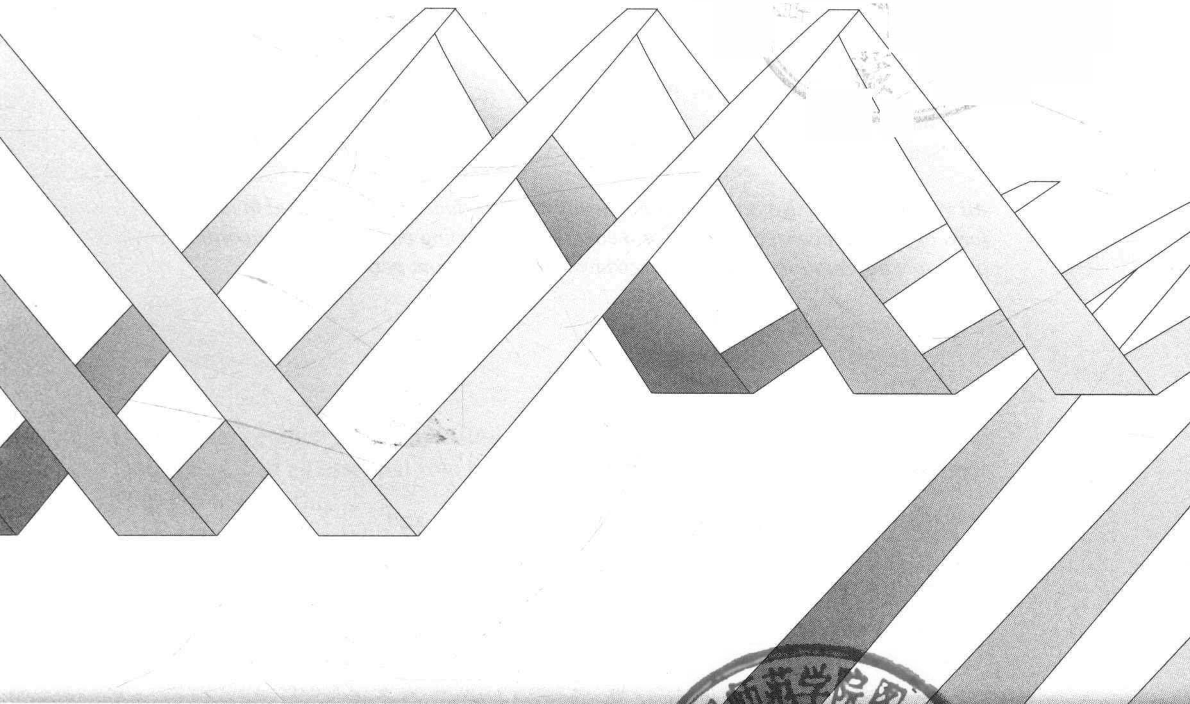

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Surveys of Modern Mathematics

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Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

With joint publication by Higher Education Press (HEP) inside China and International Press (IP) in the West with affordable prices, it is expected that books in this series will broadly reach out to the reader, in particular students, around the world, and hence contribute to mathematics and the world mathematics community.

Series Editors

Preface

Differential geometry is a major subject in contemporary mathematics. One person who had played an essential role in the rising of differential geometry is Professor Shiing-Shen Chern.

He received the Wolf prize in 1983/4 for his "outstanding contributions to global differential geometry, which have profoundly influenced all mathematics", and the citation says:

> Professor Shiing-Shen Chern has been the leading figure in global differential geometry. His earlier work on integral geometry, especially on the kinematic formula, was the source of most later work. His groundbreaking discovery of characteristic classes (now known as Chern classes) was the turning point that set global differential geometry on a course of tumultuous development. The field has blossomed under his leadership, and his results, together with those of his numerous students, have influenced the development of topology, algebraic geometry, complex manifolds, and most recently of gauge theories in mathematical physics.

In 2001, Professor Chern received a special Morningside Lifetime Achievement Award in Mathematics, and the citation reads:

> Professor Chern is awarded the Morningside Lifetime Achievement for his work on developing the foundation of Chinese mathematics, his epochal contributions to research in differential geometry, and his nurturing of leading mathematicians both in China and abroad. In the 1940s, differential geometry was at a low point worldwide; this area of mathematics was only beginning to be understood and to be used. Professor Chern became a pioneer in this subject. Some of his major achievements include the Chern characteristic classes in fiber spaces, and his proof of the Gauss-Bonnet formula. Today, differential geometry is a major subject in mathematics and a large share of the credit for this transformation goes to Professor Chern.

One natural question is how and what did such an initiator of a major subject say about the topics dear to his heart, especially when they were taking shape. What can the young generations, especially the young Chinese mathematicians, learn from a leading figure in the last century? We hope that these two books of lecture notes of Professor Chern and some of his expository papers will answer this question.

Though there are many books now on differential geometry, integral geometry and related topics, they often lack the freshness and directness of the original descriptions by masters. This is consistent with Abel's advice "By studying the masters, not their pupils." This point was endorsed by Professor Chern in his foreword to the Chinese edition of M. Atiyah's collected works:

> No matter how refined or improved a new account is, the original papers on a subject are usually more direct and to the point. When I was young, I was benefited by the advice to read Henri Poincaré, David Hilbert, Felix Klein, Adolf Hurwitz, etc. I did better with Wilhelm Blaschke, Élie Cartan and Heinz Hopf. This has also been in the Chinese tradition, when we were told to read Confucius, Han Yu in prose, and Tu Fu in poetry.

The title of the first book *Topics in Differential Geometry* comes from the title of his lecture notes at IAS, Princeton, in 1951. It also contains his expository papers: "From Triangles to Manifolds", "Curves and Surfaces in Euclidean Space", "Characteristic Classes and Characteristic Forms", "Geometry and Physics", and "The Geometry of G-Structures", together with the set of so far *unpublished lecture notes*: "Minimal Submanifolds in a Riemannian Manifold" .

Together they show how differential geometry is connected to other subjects such as topology and Lie group theory. Though there are more modern expositions of these topics, they are usually not comparable with what Chern wrote. For example, his lecture notes "Topics in Differential Geometry" starts from basics which is accessible to beginners and goes right to some striking applications such as the rigidity theorem of Cohn-Vossen on convex surfaces in $\mathbb{R}^3$ and the rigidity of convex hypersurfaces in $\mathbb{R}^n$. His treatment of the theory of connection has also such characteristic of being direct and going to the key point. The papers selected to be reprinted in this book give an overview of the scope and power of differential geometry. They complement well this set of lecture notes.

This first book will be very valuable to beginners to learn the modern differential geometry, and will also be valuable to experts for them to rethink about differential geometry.

The title of the second book is a combination of titles of two sets of lecture notes of Chern which have not been formally published. It seems that there exists only one known copy of the second set of lecture notes, which is owned by the library of University of Michigan, It also contains a more recent unpublished set of lecture notes titled "Lectures on Differential Geometry".

This second book starts with a paper of Chern which gives a gentle introduction to differential geometry accessible to students and nonexperts, and a survey of the status of global differential geometry in 1971 when Chern gave a plenary talk in ICM. As Gromov commented in his Math Review of this paper:

> This is a brilliant and inspiring exposition. The author begins with a brief historical survey, outlines some fundamental notions and tools, and describes the current situation in four branches of differential geometry: manifolds of positive curvature, curvature and the Euler characteristic, minimal sub manifolds isometric mappings, holomorphic mappings.

This paper also includes some open problems and hence is also very interesting from the historical perspective.

As every student knows, there are two key components of calculus: differentiation and integration. In geometry, there are also two closely related subjects: differential geometry and integral geometry. It is valuable to read and learn them side by side.

The lecture notes "Differential Manifolds" gives a smooth and rapid introduction to differential manifolds and differential geometry. It was delivered by Chern in 1959 at University of Chicago when differential geometry was becoming a major subject in mathematics. Its freshness shines through.

As the citation of the Wolf prize indicates, Chern also made major contribution to integral geometry. The lecture notes "Lectures on Integral Geometry" give an efficient and also accessible introduction to this subject. It is worthwhile to point out that one of Chern's teachers was Blaschke, who was a major or leading figure in integral geometry. This also adds something special to his lectures. In this set of lecture notes, one can also see Chern's global view of mathematics. For example, besides standard topics in integral geometry in Euclidean spaces, it also discusses integral geometry of homogeneous spaces.

This second book will also be a very valuable introduction to both differential and integral geometry for beginners and a supplementary reading for people at all stages.

We hope and believe that there is much one can learn from these collections of lecture notes and papers of Professor Chern, a modern master in mathematics and one of the originators of global differential geometry.

In collecting material for and editing these two books, we have received generous help from Professor Chuu-Lian Terng, and support and blessing from May Chu, the daughter of Processor Chern. Liping Wang of the Higher Education Press has also been very supportive of this project. Without their kind help, these books probably cannot appear in print. We would like thank them sincerely.

Finally, may you enjoy these two books and benefit from them!

Shiu-Yuen Cheng and Lizhen Ji

January 14, 2016.

Contents

Part I What is Geometry and Differential Geometry

1

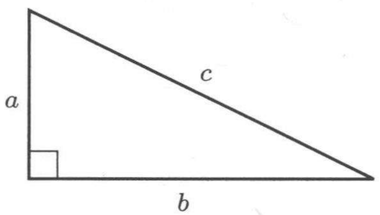
What Is Geometry? [1]

To avoid misunderstanding, I will not give a definition of geometry as in the customary mathematica treatment of a topic. I will only try to discuss its major historical developments.

1.1 Geometry as a logical system; Euclid

Euclid's "Elements of Geometry" (ca. 300 B.C.) is one of the great achievements of the human mind. It makes geometry into a deductive science and the geometrical phenomena as the logical conclusions of a system of axioms and postulates. The content is not restricted to geometry as we now understand the term. Its main geometrical results are:

a) Pythagoras' Theorem (Fig. 1).



$$c^2 = a^2 + b^2$$

Fig. 1.

b) Angle-sum of a triangle (Fig. 2).

[1] Worked done under partial support of NSF grant DMS-87-01609. The American Mathematical Monthly, Vol. 97, No. 8, Special Geometry Issue (Oct., 1990), pp. 679–686.

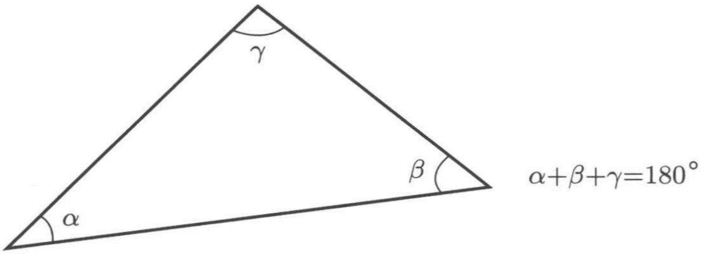


Fig. 2.

The result b) is derived using the fifth, or the last, postulate, which says: "And that, if a straight line falling on two straight lines make the angles, internal and on the same side, less than two right angles, the two straight lines, being produced indefinitely, meet on the side on which are the angles less than two right angles."(Fig. 3)

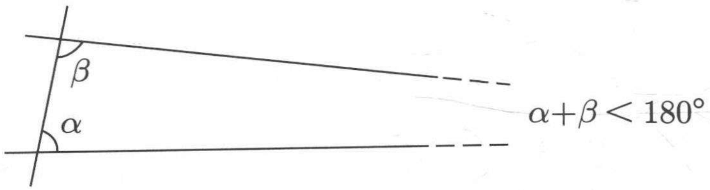


Fig. 3.

Euclid realized that the parallel postulate was not as transparent at his other axioms and postulates. Efforts were made to prove it as a consequence. Their failure led to the discovery of non-Euclidean geometry by C. F. Gauss, John Bolyai, and N. I. Lobachevski in the early 19th century.

The "Elements" treated rectilinear figures and the circles. The last three of its thirteen Books were devoted to solid geometry.

1.2 Coordinatization of space; Descartes

The introduction of coordinates by Descartes (1596–1650) was a revolution in geometry. In the plane it can be described by Fig. 4, where the role of the two coordinates x, y is not symmetric. Descartes' work was published in 1637 as an appendix, entitled "La géométrie", to his famous book on philosophy [6]. At about the same time Fermat (1601–1665) also found the concept of coordinates and used them to treat successfully geometric problems by algebraic methods. But Fermat's work was published only posthumously [7].

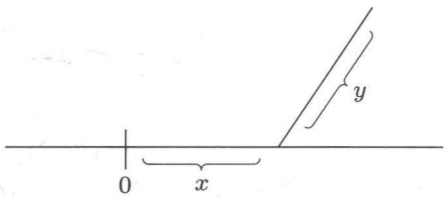


Fig. 4.

One immediate consequence was the study of curves defined by arbitrary equations

$$F(x,y) = 0, \tag{1}$$

thus enlarging the scope of the figures.

Fermat went on to introduce some of the fundamental concepts of the calculus, such as the tangent line and maxima and minima.

From two dimensions one goes to n dimensions and to an infinite number of dimensions. In these spaces one studies loci defined by arbitrary systems of equations. Thus a great vista was opened, and geometry and algebra became inseparable.

A mystery is the role of differentiation. The analytic method is most effective when the functions involved are smooth. Hence I wish to quote a philosophical question posed by Clifford Taubes [15]: Do humans really take derivatives? Can they tell the difference?

Coordinate geometry paved the way to applications to physics. An example was Newton's derivation of the Kepler laws from his law of gravitation. Kepler's first law says that the planetary orbits are ellipses with the sun as their common focus. The proof was possible only after an analytic theory of conics had been established.

1.3 Space based on the group concept; Klein's Erlanger Programm

Works on geometry led to the development of projective geometry, among whose founders wer J. V. Poncelet (1788–1867), A. F. Möbius (1790–1868), M. Chasles (1793–1880), and J. Steiner (1796–1863). Projective geometry studies the geometrical properties arising from the linear subspaces of a space and the transformations generated by projections and sections. Or the geometries resulted, the most notable ones being affine geometry and conformal geometry.

In 1872 Felix Klein formulated his Erlanger Programm [1, 11], which defines geometry as the study of the properties of a space that are invariant under a group of transformations. Thus there is a geometry corresponding to every group of transformation acting on a space. The basic notion is "group" and the notion of a space is now greatly expanded. In a certain sense the group of projective collineations is the most encompassing group and projective geometry occupies a dominant position.

The most important application of the Erlanger Programm was the treatment of non-Euclidean geometry by the so-called Cayley-Klein projective metric [12]. The hyperbolic space can be identified with the interior of a hypersphere and the non-euclidean motions with the group of projective collineations leaving invariant the hypersphere. The same group may appear as a group of transformations in different spaces. As a result the same algebraic argument could give entirely different geometric theorems. For example, everybody knows that the three medians of a triangle meet in a point. By using Study's dual numbers this translates into the following theorem of J. Petersen and F. Morley: Let ABCDEF be a skew hexagon such that consecutive sides are perpendicular. The three common perpendiculars of the pairs of opposite sides $AB, DE; BC, EF; CD, FA$ have a common perpendicular. See [13].

Sophus Lie founded a theory of general transformation groups, which became a fundamental tool of all geometry.

1.4 Localization of geometry; Gauss and Riemann

In his monograph on surface theory published in 1827 [8], Gauss (1777–1855) developed the geometry on a surface based on its fundamental form. This was generalized by B. Riemann (1826–1866) to n dimensions in his Habilitationschrift in 1854 [14]. Riemannian geometry is the geometry based on the quadratic differential form

$$ds^2 = \sum g_{ik}(u)du^i du^k, \qquad g_{ik} = g_{ki}, \qquad 1 \leq i,k \leq n \tag{2}$$

in the space of the coordinates $u^1, \ldots, u^n$, where the form is positive definite, or at least non-degenerate. Given ds^2, one can define the arc length of a curve, the angle between two interesting curves, the volume of a domain, and other geometrical concepts.

The main characteristic of this geometry is that it is local: it is valid in a neighborhood of the u-space. Because of this feature it fits well with field theory in physics. Einstein's general theory of relativity interprets the physical universe as a four-dimensional Lorentzian space (with a ds^2 of signature $+++-$) satisfying the field equations.

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi \kappa T_{ik}, \tag{3}$$

where R_{ik} is the Ricci curvature tensor, R is the scalar curvature K, is a constant, and T_{ik} is the energy-stress tensor

It is soon observed that most properties of Riemannian geometry derive from its Levi-Civita parallelism, an infinitesimal transport of the tangents paces. In other