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Entry-exit Decisions under Uncertainty

不确定环境下的进入—退出决策

张永超　张　娜　著

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Preface

Entry-exit decisions apply to numerous practical problems. For example, when to extract oil and when to stop the extraction, when to issue a new policy and when to end it, and when to let a kind of product enter a market and when to let the product exit the market, etc. Therefore, entry-exit decision problems attract a lot of researchers.

There are three approaches to study entry-exit decisions, namely, real option, pure probability and optimal stopping. In this monograph, we appeal to optimal stopping to deal with entry-exit decisions. The main reasons are as follows. On the one hand, in the real option framework, the regularity of payoff functions is *a priori* assumed, while the optimal stopping approach intends to prove it. On the other hand, although the pure probability and optimal stopping approaches are both to solve a optimal stopping problem, we have to calculate density functions of some stopping times if applying the pure probability approach, which is not easy, whereas the optimal stopping one avoids such calculations.

We aim to obtain closed-form solutions of optimal entry-exit decisions for the cases: costs depending on underlying processes, implementation with delay, and underlying processes following geometric Lévy processes. In addition, we provide a complete theory for optimal stopping problems with regime switching, and use it to solve an exit problem.

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Contents

Overview

Consider an investment as follows.

> A firm has an option to invest in a project as well as stop it. To start the project activity, the firm needs an initial investment cost to produce a single commodity at a running cost. For simplicity, let us assume that the firm produces one unit commodity per unit time when the project is active. Besides, it may also abandon the project at a terminal investment cost.

What time is optimal to enter the project and what time is optimal to exit the project? The above problem is the so-called "entry-exit decision" problem in literature.

Dixit[18] explored this problem under the framework of real option theory, assuming that the price process of one unit commodity obeys a geometric Brownian motion. He derived a system of ordinary differential equations by following a no arbitrage argument and then obtained a semi-closed solution. However, he did not prove the existence and uniqueness of the solution of the system before doing numerical analysis.

Shirakawa[40] showed a more explicit solution by employing pure probabilistic analysis under the same assumptions as Dixit's. In his discussions, density functions of some stopping times need to be calculated, which is not an easy thing. Kongsted[30] established a general deterministic limit that corresponds to Dixit's model of entry-exit decisions under uncertainty.

Dixit[18] considered some extensions without explicit calculations. Tsekrekos[43] intensively studied one of these extensions that the price process follows a geometric mean-reverting process. Another mean-reverting process used in literature is the exponential Ornstein-Uhlenbeck process. Under the assumption that the price process satisfies an exponential Ornstein-Uhlenbeck process, Levendorskiĭ[31] dealt with an entry decision problem and an exit decision problem after discussing perpetual American options.

Duckworth and Zervos[19] and Sødal[41] modeled the price process via general autonomous Itô diffusions. Duckworth and Zervos[19] allowed running payoff functions to take nonlinear forms and then addressed the problem from the programming approach. However, in order to get explicit solutions, they assumed that the initial investment cost and terminal investment cost are some constants. Sødal[41] used a discount factor approach to investment to analyse entry-exit decisions. For obtaining explicit solutions, both Duckworth and Zervos[19] and Sødal[41] assumed that the price process follows a geometric Brownian motion.

Boyarchenko and Levendorskiĭ[6] studied entry decision problems and exit decision decision problems in general Lévy process settings via the real option approach. They adopted Wiener-Hopf factorization, which is a perfect result in the probability theory, in their discussions.

Under the assumption that the price process satisfies a geometric Brownian motion whose mean and variance switch between a finite number of regimes, Hainaut[26] investigated entry-exit decisions and obtained a semi-closed solution subject to determinate times.

The references mentioned above mainly focus on different price processes. There are some other directions of extending Dixit's model, for instance, uncertain costs, multiple entry-exit decisions, investment lags, etc.

Pindyck[37] derived an entry decision rule for an irreversible investment subject to an uncertain running cost (without any initial investment cost and terminal investment cost), assuming that the value of the project is known with certainty. In some sense, Choi and Lee[13] established a more general model than Pindyck's. Then, through the real option approach, they provided a semi-closed solution of entry-exit decisions.

Brekke and Øksendal[7] explored multiple entry-exit decisions via solving an impulse control problem. They obtained an explicit solution by assuming that the price process obeys a geometric Brownian motion and the costs are some constants. Johnson and Zervos[27] intensively studied multiple entry-exit decisions. They allowed running payoff functions to take nonlinear forms. However, in order to get explicit solutions, they assumed that the initial investment cost and terminal investment cost are some constants and the price process satisfies a geometric Brownian motion.

Under the same provisos as Dixit's, investment lags were considered by Bar-Ilan and Strange[3]. They derived a system of ordinary differential equations by a no arbitrage argument and then obtained a semi-closed solution. However, they did not prove the existence and uniqueness of the solution of the system before showing numerical analysis. Applying the probabilistic approach to the entry-exit decisions with the Parisian implementation delay, Costeniuc *et al.*[15] presented an analytic solution to the optimal starting and stopping levels.

In this monograph, which is based on [45, Chapter 3] and [46–49], we study entry-exit decisions from the perspective of optimal stopping, and provide explicit solutions to them. Chapter 1 disposes of entry-exit decisions with linear costs which may be viewed as a class of uncertain costs. Implementation delay is examined in Chapter 2. In Chapter 3, we deal with the cases where underlying precesses obey geometric Lévy processes. The last chapter attends to optimal stopping problems with regime switching. We appeal to the verification argument, viscosity solution technique and Wiener-Hopf factorization to solve optimal stopping problems.

1
Entry-exit decisions with linear costs

Summary: Instead of assuming that costs are constant in classical research, we assume that they are linear with respect to the price of the commodity produced by a project. Under this assumption, we obtain a condition which guarantees that investing in the project is worthless; additionally, the project may be terminated when the commodity price is greater than a certain value. In contrast, there are no such results provided that the costs are constant. Moreover, we provide an explicit solution of entry-exit decisions if the project is worthy to be invested in.

1.1 Introduction

We do not assume that costs are constants as in classical literature, but that they are some linear functions of the commodity price. Therefore, each cost consists of two parts. One part is fixed, and the other part is proportional to the commodity price. Consequently, the costs are uncertain. As usual, we accept that the price process follows a geometric Brownian motion and, for some integrability reason, the discount rate is greater than the drift of the geometric Brownian motion.

If the costs are constant, there is always a certain time at which the firm should enter the project. However, if the costs are some linear functions of the commodity price, we obtain a condition which guarantees that the firm should never enter the project (Theorem 1.5.1 or the conclusion ① of Theorem 1.5.10). We also obtain an explicit solution of entry-exit decisions provided that the project is worthy to be invested in (the conclusions ② and ③ of Theorem 1.5.10).

Assume that the firm has already entered the project. If the costs are constant, under some provisos, the firm should exit the project when the price is less than a certain value. However, there is another possibility if the costs are linear with respect to the commodity price. That is, under some assumptions, the firm may exit the project so as to get the maximal expected profit when the price is greater than a certain value (Theorem 1.4.5 and Theorem 1.4.7). As a comparison, there is no such a result in the case that the costs are constant.

The rest of the chapter is organized as follows. In Section 1.2, we introduce an elementary theory of optimal stopping problems. In Section 1.3, the model is described in detail. In Section 1.4, we offer an optimal exit time premised on the assumption that the firm has already activated the project. In Section 1.5, we obtain an optimal entry-exit decision as to when the firm enters the project and when exits the project. Some conclusions are drawn in Section 1.6.

1.2 An elementary introduction to optimal stopping problems

Let us turn our attention to more general optimal stopping problems.

The materials in this section are mainly taken from [35, pp. 27–28] and [39, pp. 231–235]. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. We assume that $\{\mathscr{F}_t\}_{t\geq 0}$ satisfies the usual conditions and $\mathscr{F}_0$ is the completion of $\{\emptyset, \Omega\}$. Let $B = (B(t), t \geq 0)$ be a d-dimensional standard Brownian motion.

Fix an open set $\mathcal{S} \subset \mathbb{R}^{1+n}$ and let $Y = (Y(t), t \geq 0)$ be a diffusion in $\mathbb{R}^n$ given by

$$\mathrm{d}Y(t) = \alpha(t, Y(t))\mathrm{d}t + \beta(t, Y(t))\mathrm{d}B(t), \quad Y(0) = y,$$

where $\alpha : \overline{\mathbb{R}^+} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\beta : \overline{\mathbb{R}^+} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are given two functions such that a unique strong solution Y exists.

Define $T_{\mathcal{S}} := \inf\{t > 0 : (t, Y(t)) \notin \mathcal{S}\}$ and let $\mathcal{T}$ denote the set of all stopping times $\tau \le T_{\mathcal{S}}$.

For any $\tau \in \mathcal{T}$ and $y \in \mathbb{R}^n$, we define

$$J^\tau(y) := \mathbb{E}\left[\int_0^\tau f(t, Y(t))\mathrm{d}t + g(\tau, Y(\tau)) \cdot \mathbf{1}_{\{\tau<\infty\}} \middle| Y(0) = y \right],$$

where $f : \overline{\mathbb{R}^+} \times \mathbb{R}^n \to \mathbb{R}$ and $g : \overline{\mathbb{R}^+} \times \mathbb{R}^n \to \mathbb{R}$ are two functions such that the expectation $J^\tau(y)$ exits.

Then the general optimal stopping problem is described as follows.

$$\text{(1.2.1)} \qquad \text{Find } \hat{J}(y) \text{ and } \tau^* \in \mathcal{T} \text{ such that } \hat{J}(y) = \sup_{\tau\in\mathcal{T}} J^\tau(y) = J^{\tau^*}(y).$$

Now we give a procedure for solving the optimal stopping problem (1.2.1).

(1) Find a function ϕ defined on $\overline{\mathcal{S}}$ and an open set $D \subset \mathcal{S} \cap ((0, +\infty) \times \mathbb{R}^n)$ such that the following properties hold.

(1.a) The function ϕ is C^1 in $\mathcal{S}$.

(1.b) The equality $\frac{\partial}{\partial t}\phi(t, y) + \mathcal{L}\phi(t, y) + f(t, y) = 0$ holds in D, where $\mathcal{L}$ is the infinitesimal generator of the diffusion Y. Moreover, for any $(t, y) \in \partial D$ with $t > 0$, we have $\phi(t, y) = g(t, y)$.

(1.c) The inequality $\phi \ge g$ holds in $\mathcal{S}$, and the inequality $\phi > g$ holds in D.

(1.d) The inequality $\frac{\partial}{\partial t}\phi(t, y) + \mathcal{L}\phi(t, y) + f(t, y) \le 0$ holds in $\mathcal{S} \setminus \overline{D}$.

(2) Verify that the stopping time $\tau^* := \inf\{t : t > 0, (t, Y(t)) \notin D\}$ is an optimal stopping time and the function $\hat{J}$ in (1.2.1) is given by $\hat{J}(y) = \phi(0, y)$.

Remark 1.2.1 The above procedure is a modification of that in [39, pp. 234–235] or [35, p. 29, Theorem 2.2].

1.3 The model

Let $B(t)$ be a one dimensional standard Brownian motion, which denotes uncertainty in a market, defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, \mathbb{P})$. Here, $\{\mathscr{F}_t\}_{t\ge0}$ satisfies the usual conditions and $\mathscr{F}_0$ is the completion of $\{\emptyset, \Omega\}$.

We assume that the price process P follows

$$dP(t) = \mu P(t)dt + \sigma P(t)dB(t), \text{ and } P(0) > 0, \tag{1.3.1}$$

where $\mu \in \mathbb{R}$, and $\sigma > 0$.

Applying Itô's formula, we deduce that the solution of the equation (1.3.1) is

$$P(t) = P(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right]. \tag{1.3.2}$$

The firm is risk neutral. Moreover, the running cost C, the initial investment cost K_I and the terminal investment cost K_O are taken the following forms:

$$\begin{aligned} C(p) &= c_1 p + c_0, \\ K_I(p) &= k_1 p + k_0, \\ K_O(p) &= l_1 p + l_0, \end{aligned}$$

respectively. Here, c_i, k_i and l_i are some constants, $i = 0, 1$, such that $c_0 \geq 0$, $k_0 \geq 0$, $k_0 + l_0 \geq 0$ and $k_1 + l_1 \geq 0$.

To answer the questions—what time is optimal to enter (exit) the project, we will solve the optimal problem

$$\begin{aligned} \hat{J}_{IO}(p) := \sup_{0 \leq \tau_I \leq \tau_O} \mathbb{E}\Bigg[\int_{\tau_I}^{\tau_O} & \exp(-rt)(P(t) - c_1 P(t) - c_0)dt - \\ & \exp(-r\tau_I)\left(k_1 P(\tau_I) + k_0\right) - \\ & \exp(-r\tau_O)\left(l_1 P(\tau_O) + l_0\right) \Big| \, P(0) = p\Bigg], \end{aligned} \tag{1.3.3}$$

where τ_I and τ_O are stopping times, and r is the discount rate such that $r > 0$. As usual (for some integrability reason), we assume that $r > \mu$. We call the stopping times τ_I and τ_O entry times and exit times, respectively. We refer to the function $\hat{J}_{IO}$ as the maximal expected present value of the project.

1.4 An optimal exit time

In this section, we assume that the firm has already activated the project. So we may ask at what time the firm should stop the project. This problem can be solved by considering the optimal problem

$$\begin{aligned}\hat{J}_O(p) := \sup_{0\leq\tau_O} \mathbb{E}\Bigg[\int_0^{\tau_O} &\exp(-rt)((1-c_1)P(t)-c_0)\mathrm{d}t - \\ &\exp(-r\tau_O)(l_1P(\tau_O)+l_0)\,|\,P(0)=p\Bigg].\end{aligned} \tag{1.4.1}$$

We want to find a stopping time τ_O^* (called an optimal exit time) such that

$$\begin{aligned}\hat{J}_O(p) =\mathbb{E}\Bigg[\int_0^{\tau_O^*} &\exp(-rt)((1-c_1)P(t)-c_0)\mathrm{d}t - \\ &\exp(-r\tau_O^*)(l_1P(\tau_O^*)+l_0)\,|\,P(0)=p\Bigg],\end{aligned}$$

and an explicit expression of $\hat{J}_O$.

The following lemma is useful in our discussions.

Lemma 1.4.1 *The family of random variables* $\{\exp\left[(\mu-r-\sigma^2/2)\tau+\sigma B(\tau)\right] : \tau\in\mathcal{T}\}$ *is uniformly integrable, where* $\mathcal{T}$ *is the collection of all stopping times.*

Proof. Taking a real number α such that $1<\alpha<1+2(r-\mu)/\sigma^2$, we have

$$\alpha\left(\mu-r-\frac{\sigma^2}{2}\right)+\frac{\alpha^2\sigma^2}{2}<0. \tag{1.4.2}$$

Define $X(t) := \exp\left[(\mu-r-\sigma^2/2)t+\sigma B(t)\right]$ for any $t\geq 0$. Then for any $s<t$, we find that

$$\begin{aligned}\mathbb{E}\left[X(t)^\alpha|\mathscr{F}_s\right] &= \exp\left[\alpha\left(\mu-r-\frac{\sigma^2}{2}\right)t+\frac{\alpha^2\sigma^2}{2}(t-s)+\alpha\sigma B(s)\right] \\ &\leq X(s)^\alpha,\end{aligned}$$

where we have used the fact that the process

$$\left(\exp\left(-\frac{\alpha^2\sigma^2}{2}t+\alpha\sigma B(t)\right), t\geq 0\right)$$

is a martingale (see [1, p. 288, Corollary 5.2.2]) for the equality and (1.4.2) for the inequality.

Thus $(X(t)^\alpha, t\geq 0)$ is a supermartingale with a last element 0.

Then from Doob's optional sampling theorem (see [28, p. 19, Theorem 3.22] or [38, p. 9, Theorem 16]), it follows that

$$\mathbb{E}\left[X(\tau)^{\alpha}\right] \leq X(0)^{\alpha} = 1,$$

for any stopping time τ.

Therefore, according to [38, pp. 8–9, Theorem 11], the family of random variables $\left\{\exp\left[(\mu - r - \sigma^2/2)\tau + \sigma B(\tau)\right] : \tau \in \mathcal{T}\right\}$ is uniformly integrable. □

Theorem 1.4.2 *Assume that one of the following conditions*

① $l_1(r-\mu) - c_1 > -1$ *and* $c_0 \leq rl_0$,

② $l_1(r-\mu) - c_1 = -1$ *and* $c_0 < rl_0$

holds. Then the optimal exit time τ_O^* *is given by* $\tau_O^* = +\infty$ *a.s., i.e., the firm should never exit the project. Furthermore, the function* $\hat{J}_O$ *in* (1.4.1) *is given by* $\hat{J}_O(p) = \dfrac{1-c_1}{r-\mu}p - \dfrac{c_0}{r}$.

Proof. (1) Define an operator $\mathcal{L}$ by

$$\mathcal{L}\zeta(t,p) = \frac{\partial \zeta}{\partial t} + \mu p \frac{\partial \zeta}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \zeta}{\partial p^2}. \tag{1.4.3}$$

(Warning! Here, $\mathcal{L}$ is not the infinitesimal generator of the price process P.)

For any $(s,p) \in \overline{\mathbb{R}^+} \times (0,+\infty)$, set $f(s,p) := \exp(-rs)\left[(1-c_1)p - c_0\right]$ and $g(s,p) := -\exp(-rs)(l_1 p + l_0)$. Then, via a direct calculation, we find that

$$\mathcal{L}g(s,p) > -f(s,p).$$

(2) By Itô's formula, it follows that for $s \leq t$

(1.4.4)

$$g(t,P(t)) = g(s,P(s)) + \int_s^t \mathcal{L}g(u,P(u))\mathrm{d}s + \int_s^t \frac{\partial g}{\partial p}(u,P(u))\sigma P(u)\mathrm{d}B(u).$$

For a nonnegative number s, define a sequence of stopping times $(R_k(s), k \in \mathbb{N})$ by $R_k(s) := \inf\{t : t > s, P(t) > k\}$. Next, for a stopping time τ, define a sequence of stopping times $(\tau_k(s), k \in \mathbb{N})$ by $\tau_k(s) := (\tau \wedge k \wedge R_k(s)) \vee s$. Then using the optional sampling theorem (see [32, p. 53, Theorem 1.86] or [38, p. 10, Theorem 17]), we have

$$\mathbb{E}\left[\int_s^{\tau_k(s)} \frac{\partial g}{\partial p}(u,P(u))\sigma P(u)\mathrm{d}B(u) \,\middle|\, P(s) = p\right] = 0. \tag{1.4.5}$$