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Introduction to the Mathematics of Finance

金融数学引论

R. J. Williams

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

This monograph is intended as an introduction to some elements of mathematical finance. It begins with the development of the basic ideas of hedging and pricing of European and American derivatives in the discrete (i.e., discrete time and discrete state) setting of binomial tree models. Then a general discrete finite market model is defined and the fundamental theorems of asset pricing are proved in this setting. Tools from probability such as conditional expectation, filtration, (super)martingale, equivalent martingale measure, and martingale representation are all used first in this simple discrete framework. This is intended to provide a bridge to the continuous (time and state) setting which requires the additional concepts of Brownian motion and stochastic calculus. The simplest model in the continuous setting is the Black-Scholes model. For this, pricing and hedging of European and American derivatives are developed. The book concludes with a description of the fundamental theorems of asset pricing for a continuous market model that generalizes the simple Black-Scholes model in several directions.

The modern subject of mathematical finance has undergone considerable development, both in theory and practice, since the seminal work of Black and Scholes appeared a third of a century ago. The material presented here is intended to provide students and researchers with an introduction that will enable them to go on to read more advanced texts and research papers. Examples of topics for such further study include incomplete markets, interest rate models and credit derivatives.

For reading this book, a basic knowledge of probability theory at the level of the book by Chung [**10**] or D. Williams [**38**], plus for the chapters on continuous models, an acquaintance with stochastic calculus at the level of the book by Chung and Williams [**11**] or Karatzas and Shreve [**27**], is desirable. To assist the reader in reviewing this material, a summary of some of the key concepts and results relating to conditional expectation, martingales, discrete and continuous time stochastic processes, Brownian motion and stochastic calculus is provided in the appendices. In particular, the basic theory of continuous time martingales and stochastic calculus for Brownian motion should be briefly reviewed before commencing Chapter 4. Appendices C and D may be used for this purpose.

Most of the results in the main body of the book are proved in detail. Notable exceptions are results from linear programming used in Section 3.5, results used for pricing American contingent claims based on continuous models in Sections 4.7 through 4.9, and several results related to the fundamental theorems of asset pricing for the multi-dimensional Black-Scholes model treated in Chapter 5.

I benefited from reading treatments of various topics in other books on mathematical finance, including those by Pliska [**33**], Lamberton and Lapeyre [**30**], Elliott and Kopp [**15**], Musiela and Rutkowski [**32**], Bingham and Kiesel [**4**], and Karatzas and Shreve [**28**], although the treatment presented here does not parallel any one of them.

This monograph is based on lectures I gave in a graduate course at the University of California, San Diego. The material in Chapters 1–3 was also used in adapted form for part of a junior/senior-level undergraduate course on discrete models in mathematical finance at UCSD. The students in both courses came principally from mathematics and economics. I would like to thank the students in the graduate course for taking notes which formed the starting point for this monograph. Special thanks go to Nick Loehr and Amber Puha for assistance in preparing and reading the notes, and to Judy Gregg and Zelinda Collins for technical typing of some parts of the manuscript. Thanks also go to Steven Bell, Sumit Bhardwaj, Jonathan Goodman and Raphael de Santiago, for providing helpful comments on versions of the notes. Finally, I thank Bill Helton for his continuous encouragement and good humor.

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Contents

Chapter 1

Financial Markets and Derivatives

1.1. Financial Markets

A (primary) *financial market* consists of tradable securities such as stocks, bonds, currencies, commodities, or even indexes. One reason for the existence of financial markets is that they facilitate the flow of capital. For example, if a company wants to finance the building of a new production facility, it might sell shares of stock to investors, who buy the shares based on the anticipation of future rewards such as dividends or a rise in the stock price.

A variety of stochastic models is used in modeling the prices of securities. All such models face the usual trade off: more complex models typically provide a better fit to data (although there is the danger of overfitting), whereas simpler models are generally more tractable and despite their simplicity can sometimes provide useful qualitative insights. Finding a good balance between a realistic and a tractable model is part of the art of stochastic modeling.

Both discrete (in time and state), and continuous (in time and state) models will be considered here. The treatment of discrete time models is intended as a means of introducing notions such as hedging and pricing by arbitrage in a simple setting and to provide a bridge to the development in the continuous case. Binomial tree models are the most common example in the discrete case. Our treatment of discrete and continuous models is not meant to be exhaustive, but rather to provide an introduction that will enable students to go on to read more advanced material.

An *arbitrage opportunity* is an opportunity for a risk free profit. A financial market is said to be *viable* if there are no arbitrage opportunities. Typically, liquid financial markets move rapidly to eliminate arbitrage opportunities. Accordingly, in this book, attention will focus on financial market models that are viable.

1.2. Derivatives

A *derivative* or *contingent claim* is a security whose value depends on the value of some underlying security. Examples of derivatives are forward contracts, futures contracts, options, swaps, etc. (We recognize that the reader may not be familiar with all of these terms. Some of them are described in more detail below. For the other terms, or more detailed descriptions in general, we refer the interested reader to the book by Hull [**23**] or Musiela and Rutkowski [**32**]. For an in-depth, non-mathematical description of market microstructure, see Harris [**19**].) The underlying security on which a derivative is based could be a security in a financial market, such as a stock, bond, currency or commodity, but it could also be a derivative itself, such as a futures contract.

Secondary financial markets can be formed from derivatives. Some derivatives are sold on public exchanges such as the Chicago Board of Options Exchange (CBOE) the American Stock and Options Exchange (AMEX), the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME). However, many derivatives are sold over-the-counter (OTC); i.e., they are traded between individual entities — typically financial institutions and/or their corporate clients. The "derivatives market" is huge, and the OTC market is substantially bigger than the exchange traded market. An indication of the size of the OTC market is provided in reports of the Bank for International Settlements. For example, a May 2005 report [**2**] states that the amount of principal outstanding in the global over-the-counter derivatives market at the end of December 2004 was $248 trillion.

We shall now describe some examples of derivatives in a bit more detail.

A *forward contract* is an agreement to buy or sell an asset at a certain future time for a certain price. Forward contracts are traded *over the counter* (OTC). One of the common uses of forward contracts is to lock in an exchange rate for a future purchase in a foreign currency.

A *futures contract* is similar to a forward contract, but it is traded on a financial exchange. Futures contracts (or simply futures) differ from forward contracts in a number of respects. In particular, futures typically have a delivery month rather than a delivery date, and they follow a settlement procedure called *marking to market.* Briefly, in this procedure, an investor's initial deposit is adjusted on each trading day to reflect gains or losses in the futures price for that day. For further details, we refer the reader to [**23**] or [**32**]. Futures contracts are available for a wide range of commodities and financial assets. Exchanges on which futures are traded include the Chicago Board of Trade and the Chicago Mercantile Exchange.

An *option* is a contract which gives the holder of the option the right, but not the obligation, to buy or sell a given security at a given price (called the exercise price or *strike price*) within a fixed time period $[0, T]$. A *call option* gives the option holder the right to buy at the given price, whereas a *put option* gives the option holder the right to sell at the given price. A *European option* can only be exercised by the holder of the option at the expiration time T, whereas an *American option* can be exercised by the holder at any time in $[0, T]$. Exchanges on which options are traded include the Chicago Board of Options and the American Stock and Options Exchange.

Remark. Options based on stocks are usually written to cover the buying or selling of 100 shares of a stock. This is convenient, since shares of stock are usually traded in lots of 100. Most exchange traded options on stock are American-style options. Stock options on the Chicago Board Options Exchange effectively expire on the third Friday of the expiration month.

One might ask, why are options worth buying or selling? To help answer this, consider an example. To keep the description simple, consider a European call option.

Example. On January 4, 2000, a European call option on Cisco (symbol: CSCO) stock has a price of \$33. The option expires in January, and the strike price is \$70. The price of Cisco stock on January 4 is \$102.

If one bought such an option on 100 shares of Cisco, the option would cost \$3,300, and on January 21, 2000 (third Friday of January), one would have the right to buy 100 shares of Cisco at a price of \$70 per share. Suppose for simplicity that \$1 on January 4 is worth \$1 on January 21, 2000.

Scenario 1: Suppose the price of Cisco stock on January 21 is \$120 per share. This current price of the stock is called the *spot price* of the stock. The holder of the option will exercise it and make a net profit per share of $\$120 - \$70 - \$33$ (spot price of stock on January 21 − price under exercise of option − option price) and hence a net profit of $\$1,700$. This is a $\frac{1700}{33}\% = 51.5\%$ profit on the $\$3,300$ initial investment. On the other hand, if the $\$3,300$ had been directly invested in stock, the investor could have bought 32 whole shares of stock and the profit would have been $\$18 \times 32 = \$576$ on an investment of $\$102 \times 32 = \$3,264$, which is a $\frac{57600}{3264}\% = 17.6\%$ profit.

Scenario 2: Suppose the price of Cisco stock on January 21 is \$67 per share. The holder of the option will not exercise it and takes a loss of \$33 per share (the cost of the option per share) and hence a net loss of \$3,300. This is a 100% loss on the \$3,300 initial investment. On the other hand, if the \$3,300 had been invested directly in stock, the loss would have been $\$35 \times 32 = \$1,120$ or a 34.3% loss on an investment of \$3,264 in stock.

There are two main uses of derivatives, namely, *speculation and hedging*. For example, the buyer of a call option on a stock is leveraging his/her investment. The option may not cost much compared to the underlying stock, and the owner of the call option can benefit from a rise in the stock price without having to buy the stock. Of course, if the stock price goes down substantially and the option is not exercised, the buyer loses what he/she paid for the call option. In this case, the seller of the call option would benefit without necessarily having to buy the stock. Of course, if the stock price goes up substantially and the option seller does not already own the stock, then he/she will have to pay the price of buying the stock at the increased price and turning it over to the owner of the option. Options may be used to leverage and speculate in the market. Depending on how the investment is used, the downside losses can be substantial. On the other hand, options can be used to hedge risk. An investor *hedges* when he or she reduces the risk associated with his or her portfolio holdings by trading in the market to reduce exposure of the portfolio to the effect of random fluctuations in market prices. For example, an owner of stock who wants to hedge against a dramatic decrease in the stock price could buy a put option to sell the stock at a certain price, thereby reducing the risk of loss associated with such a dramatic decrease.

The focus of this book will be on the *pricing of derivative securities*. Derivatives will be referred to henceforth as contingent claims. Both European and American contingent claims will be considered here. (In this book, we restrict to contingent claims having a single payoff, cf. [**15**], rather than allowing the more general situation of [**28**], where a contingent claim can have an income stream over an interval of time.) A *European contingent claim* has a single expiration date that is the same as its payoff date. The owner of an *American contingent claim* can choose to cash in the claim at any time up until the expiration date, and the payoff may depend on when the owner chooses to cash in the claim. The main principle behind pricing of both types of contingent claims is that the price should be set in such a way that there is no opportunity for arbitrage in a market where the contingent claim and underlying securities are traded. The existence of an arbitrage free price for a contingent claim hinges on the assumption that there are no opportunities for arbitrage in the underlying (primary) securities market; i.e., that market is viable. Uniqueness of the arbitrage free price depends in the case of a European contingent claim on being able to replicate the payoff of the claim by trading in the (primary) securities market, and in the case of an American contingent claim on the seller of the claim being able to hedge the sale by trading in the market so as to have a portfolio whose value equals or exceeds the payoff of the claim at all times and equals it at some (random) time. These hypotheses will be defined precisely in the context of a model later on, and conditions under which they hold will be given.

The following further assumptions are made to simplify the presentation. There are (a) no transaction costs and (b) no dividends on stock. It is assumed that (c) the market is liquid and (d) trading by the investor does not move the market. In the discrete model framework, an investor is allowed (e) unlimited short selling of stock and (f) unlimited borrowing. In the continuous model framework, these assumptions will be modified to prevent "doubling strategies" which result in an arbitrage opportunity.

1.3. Exercise

1. Using a list of option prices (e.g., from the website for the Chicago Board Options Exchange or Yahoo Finance), perform a similar calculation to that done in the Example in the text, with Intel (symbol: INTC) in place of Cisco. For this, use the first Intel call option expiring in the current month and for which a price is listed. You should assume that a European call option for 100 shares of stock is purchased and that at expiration there are two possible scenarios for the stock price: it has gone up or down by 40% since purchase of the option.

Chapter 2

Binomial Model

In this chapter we consider a simple discrete (primary) financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [**12**] model. In the context of this model, we derive the unique arbitrage free prices for a European and an American contingent claim.

For this chapter and the next, we shall need various notions from probability and the theory of discrete time stochastic processes, including conditional expectations, martingales, supermartingales and stopping times. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices A and B. For further details, the reader is encouraged to consult Chung [**9**] or D. Williams [**38**].

Regarding general notation used throughout this book, we note that if X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that X is $\mathcal{G}$-measurable. Also, two random variables will be considered equal if they are equal almost surely, and two stochastic processes will be considered equal if they are indistinguishable (see Appendices B and C). In this chapter and the next, we restrict attention to finite sample spaces and consider only probability measures that give positive probability to each individual outcome. Consequently, in these chapters, equality of random variables and indistinguishability of discrete time stochastic processes actually entail equality surely.

2.1. Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many times $t = 0, 1, 2, \ldots, T$ (where $T < \infty$ is a positive integer and successive times are successive integers). At each of these times the values of two assets can be observed. One asset is a risky security called a stock, and the other is a riskless security called a bond.

The *bond* is assumed to yield a constant rate of return $r \geq 0$ over each time period $(t-1, t]$; and so assuming the bond is valued at \$1 at time zero, the value of the bond at time t is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \ldots, T. \tag{2.1}$$

We measure holdings in the bond in units, where the value of one unit at time t is B_t.

The *stock* price process is modeled as an exponential random walk such that S_0 is a strictly positive constant and

$$S_t = S_{t-1}\xi_t, \quad t = 1, 2, \ldots, T, \tag{2.2}$$

where $\{\xi_t, t = 1, 2, \ldots, T\}$ is a sequence of independent and identically distributed random variables with

$$P(\xi_t = u) = p = 1 - P(\xi_t = d), \tag{2.3}$$

where $p \in (0,1)$ and $0 < d < 1+r < u$. The last two conditions are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are random and strictly positive. Note that

$$S_t = S_0 \prod_{i=1}^{t} \xi_i, \quad t = 0, 1, \ldots, T. \tag{2.4}$$

(Here we assume that an empty product is defined to take the value 1.) One may represent the possible paths that S_t follows using a binary tree (see Figure 1). Note that there are only three distinct values for S_2; i.e., the two middle dots in Figure 1 have the same value for S_2. The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence S_0, S_1, S_2. For pricing and hedging of some contingent claims under the binomial model (e.g., a European contingent claim whose payoff depends only on the terminal value of the stock price or an American contingent claim whose payoff at time t depends only on the stock price at that time), one may use a so-called recombining tree in which only the distinct values of S_i are indicated (in particular, the two middle dots associated with values of S_2 in Figure 1 are identified). However, in general one needs the full path structure of the stock price process to price and hedge contingent claims; cf. Exercise 3 at the end of this chapter.

For concreteness, and without loss of generality, we assume that the probability space $(\Omega, \mathcal{F}, P)$ on which our random variables are defined is such that Ω is the finite set of 2^T possible outcomes for the values of the

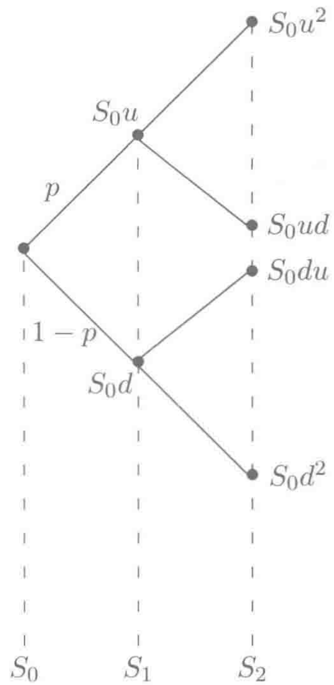

Figure 1. Binary tree for $T = 2$

stock price $(T+1)$-tuple, $(S_0, S_1, S_2, \ldots, S_T)$; $\mathcal{F}$ is the σ-algebra consisting of all possible subsets of Ω; and P is the probability measure on $(\Omega, \mathcal{F})$ associated with the Bernoulli probability p. Then, for example,

$$P((S_0, S_1, \ldots, S_T) = (S_0, S_0u, S_0u^2, \ldots, S_0u^T)) = p^T. \tag{2.5}$$

Indeed, each of the 2^T possible outcomes in Ω has strictly positive probability under P. The expectation operator under P will be denoted by $E[\cdot]$.

To describe the information available to the investor at time t, we introduce the σ-algebra generated by the stock prices up to and including time t; i.e., let

$$\mathcal{F}_t = \sigma\{S_0, S_1, \ldots, S_t\}, \quad t = 0, 1, \ldots, T. \tag{2.6}$$

In particular, with our concrete probability space, $\mathcal{F}_T = \mathcal{F}$. The family $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$ is called a *filtration*. We will often write it simply as $\{\mathcal{F}_t\}$.

A *trading strategy* (in the primary market) is a collection of pairs of random variables

$$\phi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\} \tag{2.7}$$

where the random variable α_t represents the number of shares of stock to be held over the time interval $(t-1, t]$ and the random variable β_t represents