Piotr A. Krylov, Askar A. Tuganbaev

Modules over Discrete Valuation Domains

离散赋值环上的模

De Gruyter Expositions in Mathematics 43

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by

Piotr A. Krylov and Askar A. Tuganbaev

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Introduction

There are sufficiently many books about modules over arbitrary rings (many of them are included in the bibliography). At the same time, books on modules over some specific rings are in short supply. Modules over discrete valuation domains certainly call for a special consideration, since these modules have specific properties and play an important role in various areas of algebra (especially of commutative algebra).

This book is the study of modules over discrete valuation domains. It is intended to be a first systematic account on modules over discrete valuation domains. In every part of mathematics, it is desirable to have many interesting open problems and a certain number of nice theorems. The theory presented in the book completely satisfies these conditions.

Discrete valuation domains form the class of such local domains which are very close to division rings. However, it is convenient for us to choose such a definition of a discrete valuation domain under which a division ring is not a discrete valuation domain. A discrete valuation domain is a principal ideal domain with unique (up to an invertible factor) prime element. In the theory of modules over discrete valuation domains, the role of prime elements is very important, and the nature of various constructions related to prime elements is clearly visible. Among discrete valuation domains, complete (in the p-adic topology) discrete valuation domains stand out. Typical examples of such domains are rings of p-adic integers and formal power series rings over division rings.

It is well known that all localizations of (commutative) Dedekind domains with respect to maximal ideals are discrete valuation domains. It follows from the general localization principle that it is sufficient to study many problems of the theory of modules over Dedekind domains in the case of modules over discrete valuation domains. For example, primary modules over a Dedekind domain coincide with primary modules over localizations of this ring with respect to maximal ideals.

It is necessary to emphasize close various interrelations between the theory of modules over discrete valuation domains and the theory of Abelian groups. These theories have many points of contact. This is a partial case of the principle of localization of problems, since Abelian groups coincide with modules over the ring of integers $\mathbb{Z}$ which is a commutative principal ideal domain. This implies that the theories have close ideas, methods, results, and lines of researches.

In many areas of the theory of Abelian groups, we deal with so-called *p-local* groups (i.e., modules over the localization $\mathbb{Z}_p$ of the ring $\mathbb{Z}$ with respect to the ideal generated by the prime integer p). The ring $\mathbb{Z}_p$ is a subring in the field $\mathbb{Q}$ of rational numbers; $\mathbb{Z}_p$ consists of rational numbers whose denominators are not divisible by p, and p-local groups coincide with Abelian groups G such that $G = qG$ for all prime integers $q \neq p$. With slight changes, many results on p-local groups and their proofs remain true for modules over arbitrary discrete valuation domains. In the theory of Abelian groups, modules over the ring $\widehat{\mathbb{Z}}_p$ of p-adic integers are very useful (such modules are called *p-adic modules*). The ring $\widehat{\mathbb{Z}}_p$ is the completion in the p-adic topology of the ring $\mathbb{Z}$ and the ring $\mathbb{Z}_p$. In addition, $\widehat{\mathbb{Z}}_p$ is a complete discrete valuation domain.

One of central positions in the theory of Abelian groups is occupied by p-groups, which are also called primary groups. It is appropriate to say that all modules over a fixed discrete valuation domain can be partitioned into three classes: primary modules, torsion-free modules, and mixed modules. Abelian p-groups coincide with primary $\mathbb{Z}_p$-modules as well as primary $\widehat{\mathbb{Z}}_p$-modules. We note that primary modules over discrete valuation domains are essentially presented in the literature by the theory of Abelian p-groups. With a suitable correction, main definitions, methods, and results of this theory can be transferred to primary modules. We almost are not involved in this process, since that the theory of Abelian p-groups is extensively presented in the books of Fuchs [93] and Griffith [126].

In the Kaplansky's book [166], several important theorems on modules over discrete valuation domains were included for the first time. There are three more familiar books which have appreciably affected formation of the theory of modules over discrete valuation domains. These books are the books of Baer [28] and Fuchs [92, 93]. Some topics related to the theory of endomorphism rings have their origin in the Baer's book, where the theory is developed for vector spaces (e.g., see the studies in Section 15 and Chapter 7). In the light of what has been said on the theory of Abelian groups, the reference to books of Fuchs is natural. Chapters 4, 7, and 8 of our book are related to the books of Krylov–Mikhalev–Tuganbaev [183] and Göbel–Trlifaj [109].

All main areas of the theory of modules over discrete valuation domains are presented in the book. The authors try to present main ideas, methods, and theorems which can form a basis of studies in the theory of modules over discrete valuation domains, as well as over some other rings. Some of the items presented in the book are also included in the papers [185] and [186].

Properties of vector spaces over division rings and their linear operators are assumed to be familiar; we use them without special remarks.

In comments at the end of chapters, we present some results not included in the main text and short historical remarks; we also outline other areas of studies and call attention to the literature for further examination of the field. Similar remarks are also presented in some sections. This will help to the reader to pass to the study of journal papers.

In the beginning of every chapter, we outline the content of the chapter. All sections contain exercises beginning with Section 2. Some exercises contain results from various papers. We present 34 open problems which seem to be interesting. The bibliography is quite complete, although it is possible that we did not consider some papers.

To work with the book, the reader needs to know basic results of the general theory of rings and modules. We also use certain some topological and category-theoretical ideas.

The authors assume that the book is useful to young researchers as well as experienced specialists. The book can be recommended to students and graduates studying algebra. We accept the Zermelo–Fraenkel axiomatic system from the set theory (including the choice axiom and the Zorn lemma which is equivalent to the choice axiom). The terms 'class' and 'set' are used in the ordinary set-theoretical sense. The end of the proof of some assertion is denoted by the symbol □.

The first chapter is auxiliary. In Chapter 2, we present foundations of the theory of modules over discrete valuation domains. Chapter 3 is devoted to some questions about endomorphism rings of divisible primary modules and complete torsion-free modules. In Chapter 4, we study the problem of existence of an isomorphism from an abstract ring onto the endomorphism ring of some module. In Chapter 5, torsion-free modules are studied. In Chapter 6, mixed modules are studied. In Chapter 7, we analyze the possibility of an isomorphism of two modules with isomorphic endomorphism rings. In Chapter 8, we consider several questions on transitive or fully transitive modules.

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Chapter 1

Preliminaries

In Chapter 1, we consider the following topics:

- some definitions and notation (Section 1);
- endomorphisms and homomorphisms of modules (Section 2);
- discrete valuation domains (Section 3);
- primary notions of the theory of modules (Section 4).

The first two sections contain some necessary standard information about modules. Some notation and terms are presented. In Section 2, we also consider the endomorphism ring of the module which is one of important objects of the study in the book. The material of these sections is included in the book for convenience. In Section 3 discrete valuation domains are defined and their main properties are studied. In Section 4, we lay the foundation of the theory of modules over discrete valuation domains.

1 Some definitions and notation

We assume that the reader is familiar with basic notions of the theory of rings and modules such that a ring, a module, a subring, an ideal, a submodule, the factor ring, the factor module, a homomorphism, and other notions. In the text we permanently use various elementary results on rings and modules (such as isomorphism theorems), basic properties and several constructions of rings and modules (e.g., direct sums), and some standard methods of the work with these objects. It is impossible to list all these properties. In any case, for reading the book, it is sufficient to know the theory of rings and modules within one of the three following books: F. Anderson and K. Fuller "Rings and categories of modules" ([1]), I. Lambek "Rings and modules" ([198]), F. Kasch "Modules and rings" ([170]). The considered (quite simple) category properties, can be also found in the book of S. MacLane [217]. Sometimes we touch on several aspects of the theory of topological rings and modules (e.g., see the book [2] of V. Arnautov, S. Glavatsky, and A. Mikhalev [2]). The two-volume monograph of L. Fuchs [92, 93] is an acknowledged manual in the theory of Abelian groups.

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We do present here some material related to terminology and notation. Other required definitions and results will be presented when needed.

All rings considered in the book are associative. By definition, all rings have identity elements (the only exclusions are Sections 17 and 18). The identity element is preserved under homomorphisms; it is contained in subrings and the action of the identity element on modules is the identity mapping (i.e., modules are unitary).

When we speak about an order in the set of right, left, or two-sided ideals (two-sided ideals are also called ideals), we always assume the order with respect to set-theoretical inclusion. Minimality and maximality of ideals are considered with respect to this order. Each of the three sets of ideals of the given ring forms a partially ordered set. In fact, we have three lattices. For two right (left or two-sided) ideals A and B of the ring R, the least upper bound is the intersection $A \cap B$ and the greatest lower bound is the sum $A + B$, where $A + B = \{a + b \mid a \in A,\, b \in B\}$.

If R and S are two rings and $\varphi\colon R \to S$ is a ring homomorphism, then $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}(\varphi)$ denote the image and the kernel of the homomorphism φ, respectively. The ring of all $n \times n$ matrices over the ring S with $n > 1$ is denoted by S_n.

We specialize some details related to interrelations between decompositions of rings and idempotents. A subset $\{e_1, \dots, e_n\}$ of the ring R is called a *complete orthogonal system of idempotents* if $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$, and $\sum_{i=1}^{n} e_i = 1$.

For a complete orthogonal system of idempotents, we have the Pierce decomposition of the ring into a direct sum of right ideals:

$$\{e_1, \dots, e_n\} \to R = e_1 R \oplus \cdots \oplus e_n R.$$

In addition, the right ideal $e_i R$ is not decomposable into a direct sum of right ideals if and only if the idempotent e_i is *primitive*. This means that every relation $e_i = f + g$, where f and g are orthogonal idempotents, implies that either $e_i = f$ or $e_i = g$. If we additionally assume that the idempotents e_i are *central* (therefore, they are contained in the center of the ring R), then we obtain a decomposition of the ring R into a direct sum of two-sided ideals $e_i R$. In this case, the ideal $e_i R$ is a ring with identity element e_i and the direct sum $e_1 R \oplus \cdots \oplus e_n R$ can be identified with the product of the rings $e_1 R, \dots, e_n R$. The *direct product* of some family of rings $R_1, \dots, R_n$ is denoted by

$$R_1 \times \cdots \times R_n \quad \text{or} \quad \prod_{i=1}^{n} R_i$$

(we can also write $R_1 \oplus \cdots \oplus R_n$ or $\bigoplus_{i=1}^{n} R_i$).

For a ring R, an element r of R is called a *non-zero-divisor* if $sr \neq 0$ and $rs \neq 0$ for every $0 \neq s \in R$. Otherwise, r is called a *zero-divisor*. Let R be a subring of the ring S. The ring S is called the *right classical ring of fractions* of the ring R if the following conditions hold:

(1) All non-zero-divisors of the ring R are invertible in the ring S.

(2) All elements of the ring S have the form ab^{-1}, where $a, b \in R$ and b is a non-zero-divisor of the ring R.

We say that a ring R satisfies the *right Ore condition* if for any element $a \in R$ and each non-zero-divisor $b \in R$, there exist elements $a', b' \in R$ such that b' is a non-zero-divisor and $ab' = ba'$.

It is well known that the ring R has the right classical ring of fractions if and only if it satisfies the right Ore condition. We obtain that the right classical ring of fractions of a domain with the right Ore condition is a division ring. The left classical ring of fractions and the left Ore condition are defined similarly.

Unless otherwise stated, we usually consider left modules; it was mentioned above that the modules are unitary. Similar to ideals of rings, submodules of the given module form a partially ordered set with respect to inclusion, and we also have the lattice consisting of all submodules.

We use the same notation $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ for the kernel and the image of the module homomorphism φ. If $\varphi\colon M \to N$ is a homomorphism of modules and A is a submodule of the module M, then $\varphi_{|A}$ is the *restriction* of φ to A. The restriction to A of the identity mapping of the module M is called the *embedding* from the submodule A in M.

The *direct sum* of modules A_i, $i \in I$, where I is some subscript set, is denoted by $\bigoplus_{i\in I} A_i$ or $A_1 \oplus \cdots \oplus A_n$ provided $I = \{1, 2, \ldots, n\}$. We assume that we have the direct sum $M = \bigoplus_{i\in I} A_i$. For every subscript $i \in I$, we have the coordinate embedding $\varkappa_i\colon A_i \to M$ and the coordinate projection $\pi_i\colon M \to A_i$ (details on direct sums are presented in Section 2). Setting $\varepsilon_i = \pi_i\varkappa_i$, we obtain an idempotent endomorphism of the module M, i.e., $\varepsilon_i^2 = \varepsilon_i$. Clearly, we can identify π_i and ε_i if it is convenient.

We assume that $M = A \oplus B$ and $x = a + b$, where $a \in A$ and $b \in B$. Then the elements a and b are the components of the element x. More generally, assume that either $M = \prod_{i\in I} A_i$ (the *direct product* of the family of modules A_i with $i \in I$) or $M = \bigoplus_{i\in I} A_i$. We write the element x of the module M either in the vector form $x = (\ldots, a_i, \ldots)$ or in the brief form $x = (a_i)$, where a_i is the *component* of the element x contained in the summand A_i (we also say the "coordinate").

If M is a module and n is a positive integer, then $\bigoplus_n M$ or M^n is the direct sum of n copies of the module M.

Let M be a left module over a ring R or a left R-module for brevity. For every subset X of the module M, we denote by RX the submodule of the module M generated by X. The submodule RX is the intersection of all submodules of the module M containing X. This is obvious that RX consists of all sums of the form $r_1x_1 + \cdots + r_nx_n$, where $r_i \in R$ and $x_i \in X$. If $RX = M$, then X is called a *generator system* of the module M. A module is said to be *finitely generated* if it has a finite generator system. We say that M is a *cyclic* module with generator x if $M = Rx$ for some $x \in M$.

By definition, the *sum* $\sum_{i\in I} A_i$ of the submodules A_i with $i \in I$ consists of the set of all sums of the form $a_{i_1} + \cdots + a_{i_n}$, where $a_{i_j} \in A_{i_j}$. This sum coincides with the submodule generated by the union of all submodules A_i.

If R and S are two rings, then an *R-S-bimodule* ${}_RM_S$ is an Abelian group M such that M is a left R-module and a right S-module and $(rx)s = r(xs)$ for all elements $r \in R$, $x \in M$, and $s \in S$. The ring R (more precisely, the additive group of R) can be naturally considered as a left R-module and a right R-module (these modules are also called regular modules). More precisely, we have an R-R-bimodule R.

We assume that R is a commutative ring. In this case, every left R-module M can be turned into a right R-module and conversely with the use of the relation $rx = xr$, where $r \in R$ and $x \in M$. We obtain the R-R-bimodule M.

In Section 2, we present several familiar properties of induced exact sequences of modules. Now we consider the following details. A short exact sequence of modules $0 \to A \overset{\varkappa}{\to} B \overset{\pi}{\to} C \to 0$ is said to be *split* if $B = \mathrm{Im}(\varkappa) \oplus C'$ for some module C' (we have $C' \cong C$). Every submodule A of the module M provides an exact sequence

$$0 \to A \overset{\varkappa}{\to} M \overset{\pi}{\to} M/A \to 0,$$

where $\varkappa$ is an embedding and π is the *canonical homomorphism* such that $x \to x + A$ for all $x \in M$.

Section 2 also contains main properties of the tensor product of modules which is often used in the book.

The theory of modules over discrete valuation domains and the theory of Abelian groups are congenial theories. Many sections of these theories are developed in similar ways. The reason is that discrete valuation domains and the ring of integers are close to each other, since Abelian groups and modules over the ring of integers are the same objects. Commutative discrete valuation domains and the ring of integers are contained in some special class of rings. They are examples of commutative principal ideal domains.

In some questions of the theory of modules over discrete valuation domains, categories and the category language are very useful. We give the definition of a category and consider some important related notions. An additional information about categories is considered in Sections 24 and 27. In Sections 24 and 29, we define the following three categories with a module origin: the category of quasi-homomorphisms, the Walker category, and the Warfield category. These categories will be essentially used later.

A class $\mathcal{E}$ of objects $A, B, C, \ldots$ is called a *category* if for any two objects $A, B \in \mathcal{E}$, there is a set of morphisms $\mathrm{Hom}_{\mathcal{E}}(A, B)$ with the composition

$$\mathrm{Hom}_{\mathcal{E}}(A, B) \times \mathrm{Hom}_{\mathcal{E}}(B, C) \to \mathrm{Hom}_{\mathcal{E}}(A, C)$$

such that the following two assertions hold.

(1) The composition is associative.

(2) For every object $A \in \mathcal{E}$, there exists a morphism $1_A \in \mathrm{Hom}_{\mathcal{E}}(A, A)$ such that $1_A f = f$ and $g 1_A = g$ every time, when $f \in \mathrm{Hom}_{\mathcal{E}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{E}}(B, A)$.

The morphism 1_A is called the *identity morphism* of the object A.

The category $\mathcal{E}$ is said to be *additive* if the following conditions (3) and (4) hold.

(3) For any two objects $A, B \in \mathcal{E}$, the set $\mathrm{Hom}_{\mathcal{E}}(A, B)$ is an Abelian group and the composition of morphisms is bilinear, i.e.,

$$g(f_1 + f_2) = gf_1 + gf_2 \quad \text{and} \quad (f_1 + f_2)h = f_1 h + f_2 h$$

for all

$$g \in \mathrm{Hom}_{\mathcal{E}}(C, A), \quad f_i \in \mathrm{Hom}_{\mathcal{E}}(A, B), \quad \text{and} \quad h \in \mathrm{Hom}_{\mathcal{E}}(B, D).$$

(4) There exist finite direct sums in $\mathcal{E}$. This means that for given objects

$$A_1, \ldots, A_n \in \mathcal{E},$$

there exist an object $A \in \mathcal{E}$ and morphisms $e_i \in \mathrm{Hom}_{\mathcal{E}}(A_i, A)$ such that if $f_i \in \mathrm{Hom}_{\mathcal{E}}(A_i, B)$ $(i = 1, \ldots, n)$, then there exists the unique morphism $f \in \mathrm{Hom}_{\mathcal{E}}(A, B)$ such that $e_i f = f_i$ for all $i = 1, \ldots, n$.

The object A is called the *direct sum* of objects $A_1, \ldots, A_n$ and the morphisms $e_1, \ldots, e_n$ are called *embeddings*. In this case, we write $A = A_1 \oplus \cdots \oplus A_n$ and also say that there exists a *direct decomposition* of the object A.

In the additive category $\mathcal{E}$, the set $\mathrm{Hom}_{\mathcal{E}}(A, A)$ is a ring with identity element 1_A, which is called the *endomorphism ring* of the object A; it is denoted by $\mathrm{End}_{\mathcal{E}}(A)$.

For two categories $\mathcal{C}$ and $\mathcal{E}$, the category $\mathcal{C}$ is called a *subcategory* of $\mathcal{E}$ if $\mathcal{C}$ satisfies the following conditions:

(1) All objects of the category $\mathcal{C}$ are objects of the category $\mathcal{E}$.

(2) $\mathrm{Hom}_{\mathcal{C}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{E}}(A, B)$ for any two objects $A, B \in \mathcal{C}$.

(3) The composition of any morphisms in $\mathcal{C}$ is induced by their composition in $\mathcal{E}$.

(4) All identity morphisms in $\mathcal{C}$ are identity morphisms in $\mathcal{E}$.

A subcategory $\mathcal{C}$ of the category $\mathcal{E}$ is said to be *full* if $\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{Hom}_{\mathcal{E}}(A,B)$ for any two objects $A, B \in \mathcal{C}$.

A morphism $f \in \mathrm{Hom}_{\mathcal{E}}(A,B)$ is called an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{E}}(B,A)$ such that $fg = 1_A$ and $gf = 1_B$. In this case, we say that the objects A and B are isomorphic to each other in the category $\mathcal{E}$.

We assume that we have two categories $\mathcal{E}$ and $\mathcal{D}$. A *covariant* (resp., *contravariant*) functor $F\colon \mathcal{E} \to \mathcal{D}$ from the category $\mathcal{E}$ into the category $\mathcal{D}$ consists of the mapping $\mathcal{E} \to \mathcal{D}$, $A \to F(A)$, $A \in \mathcal{E}$, and mappings

$$\mathrm{Hom}_{\mathcal{E}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B))$$

$$(\text{resp.,}\quad \mathrm{Hom}_{\mathcal{E}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),F(A))\,)$$

$(f \to F(f))$ such that they preserve the composition of morphisms and identity morphisms, i.e.,

$$F(fg) = F(f)F(g) \quad (\text{resp.,}\quad F(fg) = F(g)F(f)) \quad \text{and} \quad F(1_A) = 1_{F(A)}$$

for all objects and morphisms $A, f, g \in \mathcal{E}$.

The *identity functor* $1_{\mathcal{E}}$ of the category $\mathcal{E}$ defined by the relations

$$1_{\mathcal{E}}(A) = A \quad \text{and} \quad 1_{\mathcal{E}}(f) = f$$

for all $A, f \in \mathcal{E}$ is a covariant functor from the category $\mathcal{E}$ into $\mathcal{E}$.

Let F and G be two covariant functors from a category $\mathcal{E}$ into a category $\mathcal{D}$. A correspondence $\phi\colon F \to G$ associating a morphism $\phi_A\colon F(A) \to G(A)$ in $\mathcal{D}$ with every object $A \in \mathcal{E}$ is called a *natural transformation* $\phi\colon F \to G$ if for every morphism $f\colon A \to B$ in the category $\mathcal{E}$, we have the relation $F(f)\phi_B = \phi_A G(f)$ in the category $\mathcal{D}$. If ϕ_A is an isomorphism for every object $A \in \mathcal{E}$, then ϕ is called a *natural equivalence*. The morphism ϕ_A is called a *natural isomorphism* between $F(A)$ and $G(A)$.

One says that two categories $\mathcal{E}$ and $\mathcal{D}$ are equivalent if there exist two covariant functors

$$F\colon \mathcal{E} \to \mathcal{D} \quad \text{and} \quad G\colon \mathcal{D} \to \mathcal{E}$$

such that the functor FG (defined as the composition of the functors F and G from the left to the right) is equivalent to the identity functor $1_{\mathcal{E}}$ and the functor GF is equivalent to the identity functor $1_{\mathcal{D}}$. In this case, we say that the functors F and G define an *equivalence* of the categories $\mathcal{E}$ and $\mathcal{D}$.

In the theory of modules additive functors are usually used. A functor F is said to be *additive* if $F(f+g) = F(f) + F(g)$ for any two morphisms $f, g \in \mathcal{E}$ such that the morphism $f+g$ is defined.

2 Endomorphisms and homomorphisms of modules

We present some well known definitions and results related to endomorphism rings, group homomorphisms, and tensor products. In this section R is an arbitrary ring.

We write homomorphisms to the left side of arguments. To avoid the use of anti-isomorphic rings, we define the composition of homomorphisms as follows. Let $\alpha\colon M \to N$ and $\beta\colon N \to L$ be two homomorphisms of modules. Then the composition $\alpha\beta$ of α and β is the mapping $M \to L$ such that $(\alpha\beta)(a) = \beta(\alpha(a))$ for every $a \in M$. It is clear that the composition is a homomorphism. In some works, homomorphisms are written to the right side of the arguments, i.e., $(a)\alpha$ is used instead of $\alpha(a)$. Then for the composition $\alpha\beta$, we have

$$(a)(\alpha\beta) = ((a)\alpha)\beta, \qquad a \in M.$$

Let M and N be two R-modules. We denote by $\operatorname{Hom}_R(M, N)$ the set of all homomorphisms from the module M into the module N. (Sometimes we write "R-homomorphisms" for brevity.) The set $\operatorname{Hom}_R(M, N)$ is nonempty, since it contains the zero homomorphism $0\colon M \to N$, where $a \to 0$ for all $a \in M$. We can define the pointwise addition of homomorphisms, where

$$(\alpha + \beta)(a) = \alpha(a) + \beta(a)$$

for $\alpha, \beta \in \operatorname{Hom}_R(M, N)$ and $a \in M$. Then $\alpha + \beta$ is a homomorphism from M into N.

A homomorphism $M \to M$ is called an *endomorphism* of the module M. We set $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$. Endomorphisms can be multiplied, where the product $\alpha\beta$ coincides with the composition of the endomorphisms.

Endomorphisms of the module M which are bijections are called *automorphisms*. The identity mapping 1_M, where $1_M(a) = a$ for all $a \in M$, is an automorphism of the module M. Let $\operatorname{Aut}_R(M)$ be the set of all automorphisms of the module M. There exists the operation of multiplication of automorphisms in $\operatorname{Aut}_R(M)$.

Theorem 2.1. (a) *The set* $\operatorname{Hom}_R(M, N)$ *is an Abelian group with respect to addition of homomorphisms.*

(b) *The set* $\operatorname{End}_R(M)$ *is an associative ring with identity element.*

(c) *The set* $\operatorname{Aut}_R(M)$ *is a group with respect to multiplication of automorphisms. It coincides with the group of invertible elements of the ring* $\operatorname{End}_R(M)$.

Proof. (a) The commutativity and the associativity of addition homomorphisms are directly verified. The zero homomorphism is the zero element. For a homomorphism $\alpha\colon M \to N$, we define the homomorphism $-\alpha\colon M \to N$ by the

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