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Elementary Algebraic Geometry

代数几何学基础教程

[美] 基斯·肯迪格（Keith Kendig） 著

哈爾濱工業大學出版社
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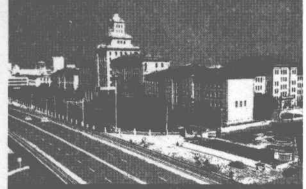

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Preface

This book was written to make learning introductory algebraic geometry as easy as possible. It is designed for the general first- and second-year graduate student, as well as for the nonspecialist; the only prerequisites are a one-year course in algebra and a little complex analysis. There are many examples and pictures in the book. One's sense of intuition is largely built up from exposure to concrete examples, and intuition in algebraic geometry is no exception. I have also tried to avoid too much generalization. If one understands the core of an idea in a concrete setting, later generalizations become much more meaningful. There are exercises at the end of most sections so that the reader can test his understanding of the material. Some are routine, others are more challenging. Occasionally, easily established results used in the text have been made into exercises. And from time to time, proofs of topics not covered in the text are sketched and the reader is asked to fill in the details.

Chapter I is of an introductory nature. Some of the geometry of a few specific algebraic curves is worked out, using a tactical approach that might naturally be tried by one not familiar with the general methods introduced later in the book. Further examples in this chapter suggest other basic properties of curves.

In Chapter II, we look at curves more rigorously and carefully. Among other things, we determine the topology of every nonsingular plane curve in terms of the degree of its defining polynomial. This was one of the earliest accomplishments in algebraic geometry, and it supplies the initiate with a straightforward and very satisfying result.

Chapter III lays the groundwork for generalizing some of the results of plane curves to varieties of arbitrary dimension. It is essentially a chapter on commutative algebra, looked at through the eyeglasses of the geometer.

Algebraic ideas are supplied with geometric meaning, so that in a sense one obtains a "dictionary" between commutative algebra and algebraic geometry. I have put this dictionary in the form of a diagram of lattices; this approach does seem to neatly tie together a good many results and easily suggests to the reader a number of possible analogues and extensions.

Chapter IV is devoted to a study of algebraic varieties in $\mathbb{C}^n$ and $\mathbb{P}^n(\mathbb{C})$ and includes a geometric treatment of intersection multiplicity (which we use to prove Bézout's theorem in n dimensions).

In Chapter V we look at varieties as underlying objects upon which we do mathematics. This includes evaluation of elements of the variety's function field (that is, a study of valuation rings), a translation of the fundamental theorem of arithmetic to a nonsingular curve-theoretic setting (the classical ideal theory), some function theory on curves (a generalization of certain basic facts about functions meromorphic on the Riemann sphere), and finally the Riemann–Roch theorem on a curve (which ties in function theory on a curve with the topology of the curve).

After the reader has finished this book, he should have a foundation from which he can continue in any of several different directions—for example, to a further study of complex algebraic varieties, to complex analytic varieties, or to the scheme-theoretic treatments of algebraic geometry which have proved so fruitful.

It is a pleasure to acknowledge the help given to me by various students who have read portions of the book; I also want to thank Frank Lozier for critically reading the manuscript, and Basil Gordon for all his help in reading the galleys. Thanks are also due to Mary Blanchard for her excellent job in typing the original draft, to Mike Ludwig who did the line drawings, and to Robert Janusz who did the shaded figures. I especially wish to express my gratitude to my wife, Joan, who originally encouraged me to write this book and who was an invaluable aid in preparing the final manuscript.

Keith Kendig

Cleveland, Ohio

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CHAPTER I
Examples of curves

1 Introduction

The principal objects of study in algebraic geometry are algebraic varieties. In this introductory chapter, which is more informal in nature than those that follow, we shall define algebraic varieties and give some examples; we then give the reader an intuitive look at a few properties of a special class of varieties, the "complex algebraic curves." These curves are simpler to study than more general algebraic varieties, and many of their simply-stated properties suggest possible generalizations. Chapter II is essentially devoted to proving some of the properties of algebraic curves described in this chapter.

Definition 1.1. Let k be any field.

(1.1.1) The set $\{(x_1, \ldots, x_n) \mid x_i \in k\}$ is called **affine n-space over k**; we denote it by k^n, or by $k_{X_1, \ldots, X_n}$. Each n-tuple of k^n is called a **point**.

(1.1.2) Let $k[X_1, \ldots, X_n] = k[X]$ be the ring of polynomials in n indeterminants $X_1, \ldots, X_n$, with coefficients in k. Let $p(X) \in k[X] \backslash k$. The set

$$\mathsf{V}(p) = \{(x) \in k^n \mid p(x) = 0\}$$

is called a **hypersurface** of k^n, or an **affine hypersurface**.

(1.1.3) If $\{p_\alpha(X)\}$ is any collection of polynomials in $k[X]$, the set

$$\mathsf{V}(\{p_\alpha\}) = \{(x) \in k^n \mid \text{each } p_\alpha(x) = 0\}$$

is called an **algebraic variety** in k^n, and **affine algebraic variety**, or, if the context is clear, just a **variety**. If we wish to make explicit reference to the field k, we say **affine variety over k**, **k-variety**, etc.; k is called the **ground field**. We also say $\mathsf{V}(\{p_\alpha\})$ **is defined by** $\{p_\alpha\}$.

(1.1.4) k^2 is called the **affine plane**. If $p \in k[X_1, X_2] \backslash k$, $\mathsf{V}(p)$ is called a **plane affine curve** (or **plane curve**, **affine curve**, **curve**, etc., if the meaning is clear from context)

We will show later on, in Section III,3, that any variety can be defined by only finitely many polynomials p_α.

Here are some examples of varieties in $\mathbb{R}^2$.

EXAMPLE 1.2

(1.2.1) Any variety $\mathsf{V}(aX^2 + bXY + cY^2 + dX + eY + f)$ where $a, \ldots, f \in \mathbb{R}$. Hence all circles, ellipses, parabolas, and hyperbolas are affine algebraic varieties; so also are all lines.

(1.2.2) The "cusp" curve $\mathsf{V}(Y^2 - X^3)$; see Figure 1.

(1.2.3) The "alpha" curve $\mathsf{V}(Y^2 - X^2(X + 1))$; see Figure 2.

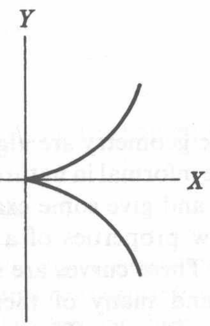

Figure 1

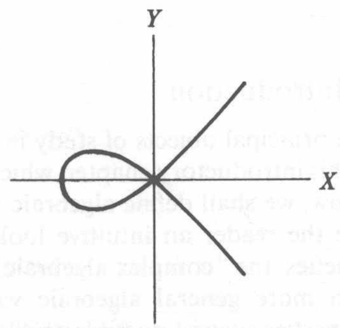

Figure 2

(1.2.4) The cubic $\mathsf{V}(Y^2 - X(X^2 - 1))$; see Figure 3. This example shows that algebraic curves in $\mathbb{R}^2$ need not be connected.

(1.2.5) If $\mathsf{V}(p_1)$ and $\mathsf{V}(p_2)$ are varieties in $\mathbb{R}^2$, then so is $\mathsf{V}(p_1) \cup \mathsf{V}(p_2)$; it is just $\mathsf{V}(p_1 \cdot p_2)$, as the reader can check directly from the definition. Hence one has a way of manufacturing all sorts of new varieties. For instance, $(X^2 + Y^2 - 1)(X^2 + Y^2 - 4) = 0$ defines the union of two concentric circles (Figure 4).

(1.2.6) The graph $\mathsf{V}(Y - p(X))$ in $\mathbb{R}^2$ of any polynomial $Y = p(X) \in \mathbb{R}[X]$ is also an algebraic variety.

(1.2.7) If $p_1, p_2 \in \mathbb{R}[X, Y]$, then $\mathsf{V}(p_1, p_2)$ represents the simultaneous solution set of two polynomial equations. For instance, $\mathsf{V}(X, Y) = \{(0, 0)\} \subseteq \mathbb{R}^2$, while $\mathsf{V}(X^2 + Y^2 - 1, X - Y)$ is the two-point set

$$\left\{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right\}$$

in $\mathbb{R}^2$.

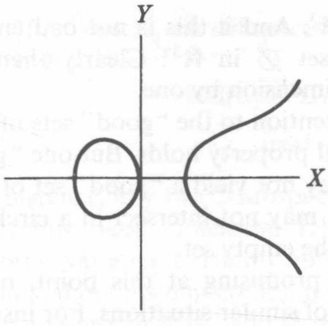

Figure 3

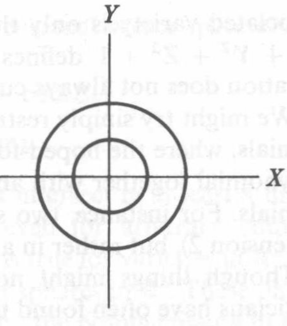

Figure 4

(1.2.8) In $\mathbb{R}^3$, any conic is an algebraic variety, examples being the sphere $V(X^2 + Y^2 + Z^2 - 1)$, the cylinder $V(X^2 + Y^2 - 1)$, the hyperboloid $V(X^2 - Y^2 - Z^2 - 1)$, and so on. A circle in $\mathbb{R}^3$ is also a variety, being represented, for example, as $V(X^2 + Y^2 + Z^2 - 1, X)$ (geometrically the intersection of a sphere and the (Y, Z)-plane). Any point (a, b, c) in $\mathbb{R}^3$ is the variety $V(X - a, Y - b, Z - c)$ (geometrically, the intersection of the three planes $X = a$, $Y = b$, and $Z = c$).

Now suppose (still using $k = \mathbb{R}$) that we have written down a large number of sets of polynomials, and that we have sketched their corresponding varieties in $\mathbb{R}^n$. It is quite natural to look for some regularity. How do algebraic varieties behave? What are their basic properties?

First, perhaps a simple "dimensionality property" might suggest itself. For our immediate purposes, we may say that $V \subset \mathbb{R}^n$ has dimension d if V contains a homeomorph of $\mathbb{R}^d$, and if V is the disjoint union of finitely many homeomorphs of $\mathbb{R}^i$ $(i \leqslant d)$. Then in all examples given so far, each equation introduces one restriction on the dimension, so that each variety defined by one equation has dimension one less than the surrounding space—i.e., the variety has *codimension* 1. (In k^n, "codimension" means "n − dimension.") And each variety defined by two (essentially different) equations has dimension two less than the surrounding (or "ambient") space (codimension 2), etc. Hence the sphere $V(X^2 + Y^2 + Z^2 - 1)$ in $\mathbb{R}^3$ has dimension $3 - 1 = 2$, the circle $V(X^2 + Y^2 + Z^2 - 1, X)$ in $\mathbb{R}^3$ has dimension $3 - 2 = 1$, and the point $V(X - a, Y - b, Z - c)$ in $\mathbb{R}^3$ has dimension $3 - 3 = 0$. This same thing happens in $\mathbb{R}^n$ with homogeneous linear equations—each new linearly independent equation cuts down the dimension of the resulting subspace by one.

But if we look down our hypothetical list a bit further, we come to the polynomial $X^2 + Y^2$; $X^2 + Y^2$ defines only the Z-axis in $\mathbb{R}^3$. This one equation cuts down the dimension of $\mathbb{R}^3$ by two—that is, the Z-axis has codimension two in $\mathbb{R}^3$. And further down the list we see $X^2 + Y^2 + Z^2$; the associated variety is only the origin in $\mathbb{R}^3$. And if this is not bad enough, $X^2 + Y^2 + Z^2 + 1$ defines the empty set $\varnothing$ in $\mathbb{R}^3$! Clearly then, one equation does not always cut down the dimension by one.

We might try simply restricting our attention to the "good" sets of polynomials, where the hoped-for dimensional property holds. But one "good" polynomial together with another one may not yield a "good" set of polynomials. For instance, two spheres in $\mathbb{R}^3$ may not intersect in a circle (codimension 2), but rather in a point, or in the empty set.

Though things might not look very promising at this point, mathematicians have often found their way out of similar situations. For instance, mathematicians of antiquity thought that only certain nonconstant polynomials in $\mathbb{R}[X]$ had zeros. But the exceptional status of polynomials having only real roots was removed once the field $\mathbb{R}$ was extended to its algebraic completion, $\mathbb{C}$ = field of complex numbers. One then had a most beautiful and central result, the fundamental theorem of algebra. (Every nonconstant polynomial $p(X) \in \mathbb{C}[X]$ has a zero, and the number of these zeros, when counted with multiplicity, is the degree of $p(X)$.) Similarly, geometers could remove the exceptional behavior of "parallel lines" in the Euclidean plane once they completed it in a geometric way by adding "points at infinity," arriving at the *projective completion* of the plane. One could then say that any two different lines intersect in exactly one point, and there was born a beautiful and symmetric area of mathematics, namely projective geometry.

For us, we may find a way out of our difficulties by using both kinds of completions. We first complete algebraically, using $\mathbb{C}$ instead of $\mathbb{R}$ (each set of polynomials $p_1, \ldots, p_r$ with real or complex coefficients defines a variety $\mathsf{V}(p_1, \ldots, p_r)$ in $\mathbb{C}^n$); and we also complete $\mathbb{C}^n$ *projectively* to *complex projective n-space*, denoted $\mathbb{P}^n(\mathbb{C})$. The variety $\mathsf{V}(p_1, \ldots, p_r)$ in $\mathbb{C}^n$ will be extended in $\mathbb{P}^n(\mathbb{C})$ by taking its topological closure. (We shall explain this further in a moment.) By extending our space and variety this way, we shall see that all exceptions to our "dimensional relation" will disappear, and algebraic varieties will behave just like subspaces of a vector space in this respect.

Hence, although in $\mathbb{R}^2$, $X^2 + Y^2 - 1$ defines a circle but $X^2 + Y^2$ only a point and $X^2 + Y^2 + 1$ the empty set, in our new setting each of these polynomials turns out to define a variety of (complex) codimension one in $\mathbb{P}^2(\mathbb{C})$, independent of what the "radius" of the circle might be. (The "complex dimension" of a variety V in $\mathbb{C}^n$ is just one-half the dimension of V considered as a real point set; we shall see later that as a real point set, the dimension is always even. Also, even though the locus in $\mathbb{C}^2$ of $X^2 + Y^2 = 1$ does not turn out to look like a circle, we shall continue to use this term since the $\mathbb{C}^2$-locus is defined by the same equation. Similarly, we shall use terms like *curve* or *surface* for complex varieties of complex dimension 1 and 2, respectively.)

In general, any nonconstant polynomial turns out to define a point set of complex codimension one in $\mathbb{P}^n(\mathbb{C})$, just as one (nontrivial) linear equation does in any vector space. A generalization of this vector space property is:

If L_1 and L_2 are subspaces of any n-dimensional vector space k^n over k, then

$$\operatorname{cod}(L_1 \cap L_2) \leqslant \operatorname{cod}(L_1) + \operatorname{cod}(L_2)$$

$$(\operatorname{cod} = \text{codimension}).$$

For instance, any two 2-subspaces in $\mathbb{R}^3$ must intersect in at least a line. In $\mathbb{P}^n(\mathbb{C})$ this basic dimension relation holds even for arbitrary complex-algebraic varieties. Certainly nothing like this is true for varieties in $\mathbb{R}^2$. One can talk about disjoint circles in $\mathbb{R}^2$, or disjoint spheres in $\mathbb{R}^3$. These phrases make no sense in $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{P}^3(\mathbb{C})$, respectively; the points missing in $\mathbb{R}^2$ or $\mathbb{R}^3$ simply are not seen because they are either "at infinity," or have complex coordinates. (This will be made more precise soon.) Hence it turns out that what we see in $\mathbb{R}^n$ is just the tip of an iceberg—a rather unrepresentative slice of the variety at that—whose "true" life, from the algebraic geometer's viewpoint, is lived in $\mathbb{P}^n(\mathbb{C})$.

2 The topology of a few specific plane curves

Suppose we have added the missing "points at infinity" to a complex algebraic variety in $\mathbb{C}^n$, thus getting a variety in $\mathbb{P}^n(\mathbb{C})$. It is natural to wonder what the entire "completed" curve looks like. We consider here only curves in $\mathbb{C}^2$ and in $\mathbb{P}^2(\mathbb{C})$; complex varieties of higher dimension have real dimension $\geqslant 4$ and our visual appreciation of them is necessarily limited. Even our complex curves live in real 4-space; our situation is somewhat analogous to an inhabitant of "Flatland" who lives in $\mathbb{R}^2$, when he attempts to visualize an ordinary sphere in $\mathbb{R}^3$. He can, however, see 2-dimensional slices of the sphere. Now in $X^2 + Y^2 + Z^2 = 1$, substituting a specific value Z_0 for Z yields the part of the sphere in the plane $Z = Z_0$. Then if he lets $Z = T =$ time, he can "visualize" the sphere by looking at a succession of parallel plane slices $X^2 + Y^2 = 1 - T^2$ as T varies. He sees a "moving picture" of the sphere; it is a point when $T = -1$, growing to ever larger circles, reaching maximum diameter at $T = 0$, then diminishing to a point when $T = 1$.

Our situation is perhaps even more strictly analogous to his problem of visualizing something like a "warped circle" in 3-space (Figure 5). The

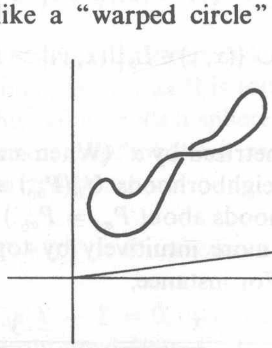

Figure 5

Flatlander's moving picture of the circle's intersections with the planes Z = constant will trace out a topological circle for him. He may not appreciate all the twisting and warping that the circle has in $\mathbb{R}^3$, but he can see its topological structure.

To get a topological look at our complex curves, let us apply this same idea to a hypersurface in complex 2-space. In $\mathbb{C}^2$, we will let the complex X-variable be $X = X_1 + iX_2$; similarly, $Y = Y_1 + iY_2$. We will let X_2 vary with time, and our "screen" will be real (X_1, Y_1, Y_2)-space. The intersection of the 3-dimensional hyperplane X_2 = constant with the real 2-dimensional variety will in general be a real curve; we will then fit these curves together in our own 3-space to arrive at a 2-dimensional object we can visualize. As with the Flatlander, we will lose some of the warping and twisting in 4-space, but we will nonetheless get a faithful topological look, which we will be content with for now.

Since our complex curves will be taken in $\mathbb{P}^2(\mathbb{C})$, we first describe intuitively the little we need here in the way of projective completions. Our treatment is only topological here, and will be made fuller and more precise in Chapter II. We begin with the real case.

$\mathbb{P}^1(\mathbb{R})$: As a topological space, this is obtained by adjoining to the topological space $\mathbb{R}$ (with its usual topology) an "infinite" point, say P, together with a neighborhood system about P. For basic open neighborhoods we take

$$U_N(P) = \{P\} \cup \{r \in \mathbb{R} \mid |r| > N\} \qquad N = 1, 2, 3, \ldots.$$

We can visualize this more easily by shrinking $\mathbb{R}^1$ down to an open line segment, say by $x \to x/(1 + |x|)$. We may add the point at infinity by adjoining the two end points to the line segment and identifying these two points. In this way $\mathbb{P}^1(\mathbb{R})$ becomes, topologically, an ordinary circle.

$\mathbb{P}^2(\mathbb{R})$: First note that, except for $\mathbb{R}_X$, the 1-spaces $L_\alpha = \mathsf{V}(X + \alpha Y)$ of $\mathbb{R}_{XY}$ are parametrized by α; a different parametrization, $L_{\alpha'} = \mathsf{V}(\alpha' X + Y)$, includes $\mathbb{R}_X$ (but not $\mathbb{R}_Y$). Then as a topological space, $\mathbb{P}^2(\mathbb{R})$ is obtained from $\mathbb{R}^2$ by adjoining to each 1-subspace of $\mathbb{R}^2$, a point together with a neighborhood system about each such point.

If, for instance, a given line is L_{α_0}, then for basic open neighborhoods about a given P_{α_0} we take

$$U_N(P_{\alpha_0}) = \bigcup_{|\alpha - \alpha_0| < 1/N} (\{P_\alpha\} \cup \{(x, y) \in L_\alpha \mid |(x, y)| > N\}) \qquad N = 1, 2, 3, \ldots,$$

where $|(x, y)| = |x| + |y|$.

Similarly for lines parametrized by α'. (When α and α' both represent the same line $L_{\alpha_0} = L_{\alpha'_0}$, the neighborhoods $U_N(P_{\alpha_0})$ and $U_N(P_{\alpha'_0})$ generate the same set of open neighborhoods about $P_{\alpha_0} = P_{\alpha'_0}$.)

Again, we can see this more intuitively by topologically shrinking $\mathbb{R}^2$ down to something small. For instance,

$$(x, y) \to \left(\frac{x}{1 + \sqrt{x^2 + y^2}}, \frac{y}{1 + \sqrt{x^2 + y^2}}\right)$$

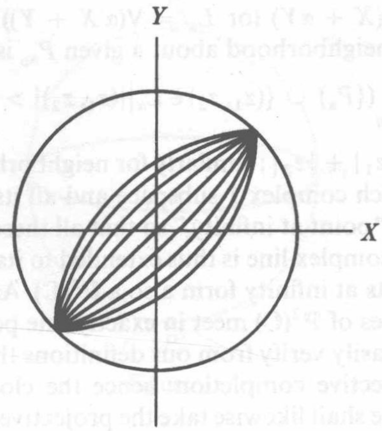

Figure 6

maps $\mathbb{R}^2$ onto the unit open disk. Figure 6 shows this condensed plane together with some mutually parallel lines. (Two lines parallel in $\mathbb{R}^2$ will converge in the disk since distance becomes more "concentrated" as we approach its edge; the two points of convergence are opposite points. If, as in $\mathbb{P}^1(\mathbb{R})$, we identify these points, then any two "parallel" lines in the figure will intersect in that one point. Adding analogous points for every set of parallel lines in the plane means adding the whole boundary of the disk, with opposite (or *antipodal*) points identified. All these "points at infinity" form the "line at infinity," itself topologically a circle, hence a projective line $\mathbb{P}^1(\mathbb{R})$. Since this line at infinity intersects every other line in just one point, it is clear that any two different projective lines of $\mathbb{P}^2(\mathbb{R})$ meet in precisely one point.

$\mathbb{P}^1(\mathbb{C})$: Topologically, the "complex projective line" is obtained by adjoining to $\mathbb{C}$ an "infinite" point P; for basic open neighborhoods about P, take

$$U_N\{P\} = \{P\} \cup \{z \in \mathbb{C} \,\big|\, |z| > N\} \qquad N = 1, 2, 3, \ldots.$$

Intuitively, shrink $\mathbb{C}$ down so it is an open disk, which topologically is also a sphere with one point missing (just as $\mathbb{R}$ is topologically a circle with one point missing). Adding this point yields a sphere.

$\mathbb{P}^2(\mathbb{C})$: As in the real case, except for the X-axis $\mathbb{C}_X$, the complex 1-spaces of $\mathbb{C}^2 = \mathbb{C}_{XY}$ are parametrized by α:

$$X + \alpha Y = 0 \quad \text{where } \alpha \in \mathbb{C};$$

another parametrization, $\alpha' X + Y = 0$, includes $\mathbb{C}_X$ but not $\mathbb{C}_Y$. Then $\mathbb{P}^2(\mathbb{C})$ as a topological space is obtained from $\mathbb{C}^2$ by adjoining to each complex

1-subspace $L_\alpha = \mathrm{V}(X + \alpha Y)$ (or $L_{\alpha'} = \mathrm{V}(\alpha' X + Y)$) a point P_α (or $P_{\alpha'}$). A typical basic open neighborhood about a given P_{α_0} is

$$U_N(P_{\alpha_0}) = \bigcup_{|\alpha - \alpha_0| < 1/N} (\{P_\alpha\} \cup \{(z_1, z_2) \in L_\alpha \,|\, |(z_1, z_2)| > N\}) \qquad N = 1, 2, 3, \ldots,$$

where $|(z_1, z_2)| = |z_1| + |z_2|$; similarly for neighborhoods about points $P_{\alpha_0'}$.

Intuitively, to each complex 1-subspace and all its parallel translates, we are adding a single "point at infinity," so that all these parallel lines intersect in one point. Each complex line is thus extended to its projective completion, $\mathbb{P}^1(\mathbb{C})$; and all points at infinity form also a $\mathbb{P}^1(\mathbb{C})$. As in $\mathbb{P}^2(\mathbb{R})$, any two different projective lines of $\mathbb{P}^2(\mathbb{C})$ meet in exactly one point.

The reader can easily verify from our definitions that each of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{C}^2$ is dense in its projective completion; hence the closure of $\mathbb{C}^2$ in $\mathbb{P}^2(\mathbb{C})$ is $\mathbb{P}^2(\mathbb{C})$, and so on. We shall likewise take the projective extension of a complex algebraic curve in $\mathbb{C}^2$ to be its topological closure in $\mathbb{P}^2(\mathbb{C})$.

We next consider some examples of projective curves using the slicing method outlined above.

EXAMPLE 2.1. Consider the circle $\mathrm{V}(X^2 + Y^2 - 1)$. Let $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$. Then $(X_1 + iX_2)^2 + (Y_1 + iY_2)^2 = 1$. Expanding and equating real and imaginary parts gives

$$X_1{}^2 - X_2{}^2 + Y_1{}^2 - Y_2{}^2 = 1, \qquad X_1X_2 + Y_1Y_2 = 0. \tag{1}$$

We let X_2 play the role of time; we start with $X_2 = 0$. The part of our complex circle in the 3-dimensional slice $X_2 = 0$ is then given by

$$X_1{}^2 + Y_1{}^2 - Y_2{}^2 = 1, \qquad Y_1Y_2 = 0. \tag{2}$$

The first equation defines a hyperboloid of one sheet; the second one, the union of the (X_1, Y_1)-plane and the (X_1, Y_2)-plane (since $Y_1 \cdot Y_2 = 0$ implies $Y_1 = 0$ or $Y_2 = 0$). The locus of the equations in (2) appears in Figure 7. It is

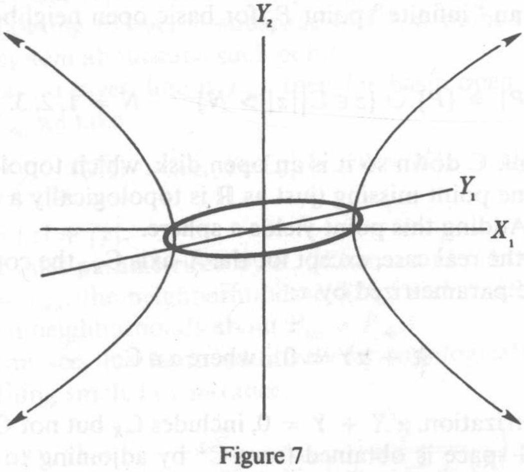

Figure 7

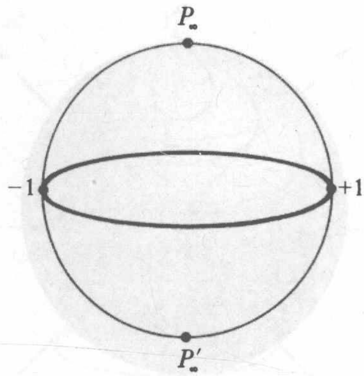

Figure 8

the union of the real circle $X_1{}^2 + Y_1{}^2 = 1$ (when $Y_2 = 0$) and the hyperbola $X_1{}^2 - Y_2{}^2 = 1$ (when $Y_1 = 0$). The circle is, of course, just the real part of the complex circle. The hyperbola has branches approaching two points at infinity, which we call P_∞ and P'_∞.

Now the completion in $\mathbb{P}^2(\mathbb{R})$ of the hyperbola is topologically an ordinary circle. Hence the total curve in our slice $X_2 = 0$ is topologically two circles touching at two points; this is drawn in Figure 8. The more lightly-drawn circle in Figure 8 corresponds to the (lightly-drawn) hyperbola in Figure 7.

Now let's look at the situation when "time" X_2 changes a little, say to $X_2 = \varepsilon > 0$. This defines the corresponding curve

$$X_1{}^2 + Y_1{}^2 - Y_2{}^2 = 1 + \varepsilon^2, \qquad \varepsilon X_1 + Y_1 Y_2 = 0.$$

The first surface is still a hyperboloid of one sheet; the second one, for ε small, in a sense "looks like" the original two planes. The intersection of these two surfaces is sketched in Figure 9. The circle and hyperbola have split into two disjoint curves. We may now sketch these disjoint curves in on Figure 8; they always stay close to the circle and hyperbola. If we fill in all

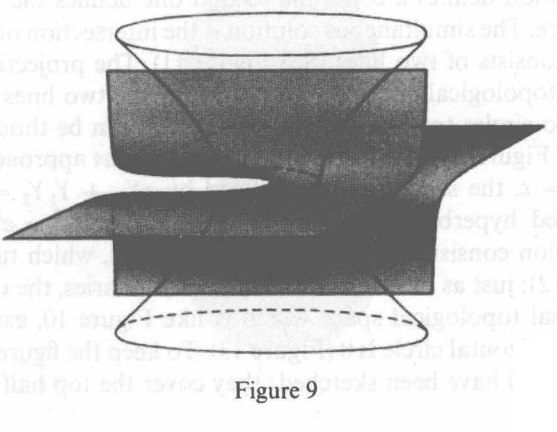

Figure 9

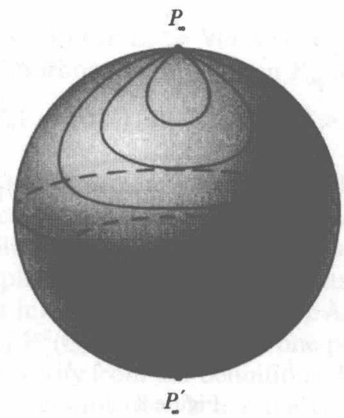

Figure 10

such curves corresponding to $X_2 =$ constant, we will fill in the surface of a sphere. The curves for nonnegative X_2 are indicated in Figure 10.

For $X_2 < 0$, one gets curves lying on the other two quarters of the sphere. We thus see (and will rigorously prove in Section II,10) that all these curves fill out a sphere. We thus have the remarkable fact that *the complex circle* $\mathsf{V}(X^2 + Y^2 - 1)$ *in* $\mathbb{P}^2(\mathbb{C})$ *is topologically a sphere.*

From the complex viewpoint, the complex circle still has codimension 1 in its surrounding space.

EXAMPLE 2.2. Now let us look at a circle of "radius 0," $\mathsf{V}(X^2 + Y^2)$. The equations corresponding to (1) are

$$X_1{}^2 - X_2{}^2 + Y_1{}^2 - Y_2{}^2 = 0, \qquad X_1X_2 + Y_1Y_2 = 0. \tag{3}$$

The part of this variety lying in the 3-dimensional slice $X_2 = 0$ is then given by

$$X_1{}^2 + Y_1{}^2 - Y_2{}^2 = 0, \qquad Y_1Y_2 = 0. \tag{4}$$

The first equation defines a cone; the second one defines the union of two planes as before. The simultaneous solution is the intersection of the cone and planes. This consists of two lines (See Figure 11). The projective closure of each line is a topological circle, so the closure of the two lines in this figure consists of two circles touching at one point. This can be thought of as the limit figure of Figure 8 as the horizontal circle's radius approaches zero.

When $X_2 = \varepsilon$, the saddle-surface defined by $\varepsilon X_1 + Y_1Y_2 = 0$ intersects the one-sheeted hyperboloid given by $X_1{}^2 + Y_1{}^2 - Y_2{}^2 = \varepsilon^2$. As before, their intersection consists of two disjoint real curves, which turn out to be lines (Figure 12); just as in the first example, as X_2 varies, the curves fill out a 2-dimensional topological space which is like Figure 10, except that the radius of the horizontal circle is 0 (Figure 13). To keep the figure simple, only curves for $X_2 \geqslant 0$ have been sketched; they cover the top half of the upper