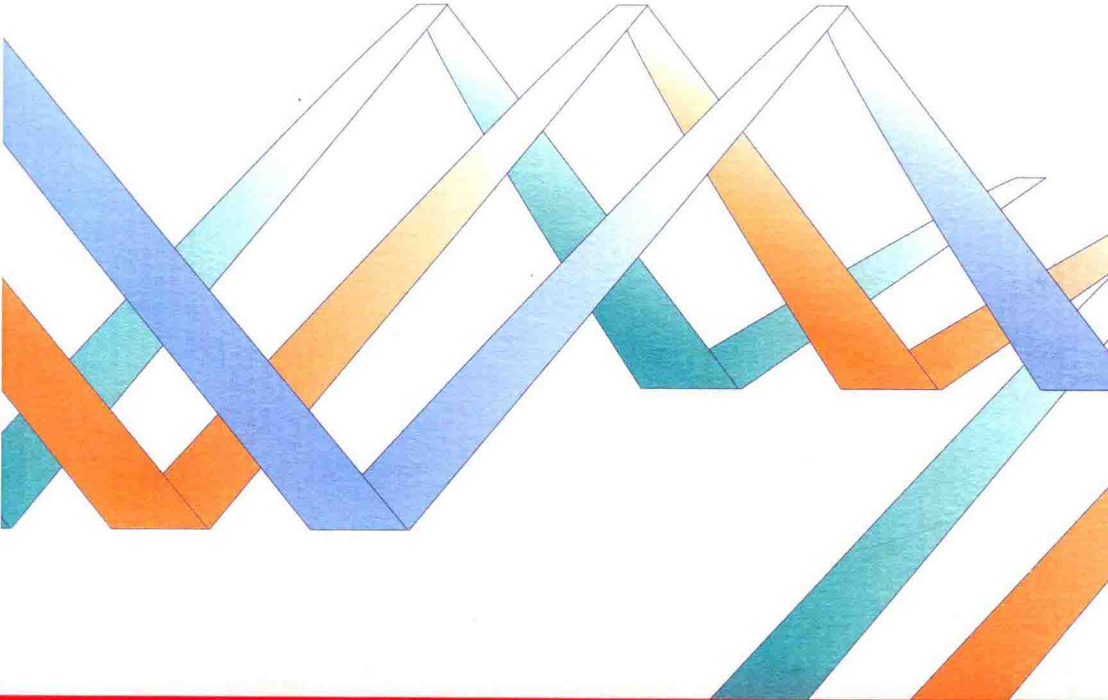
SMM 11
Surveys of Modern Mathematics



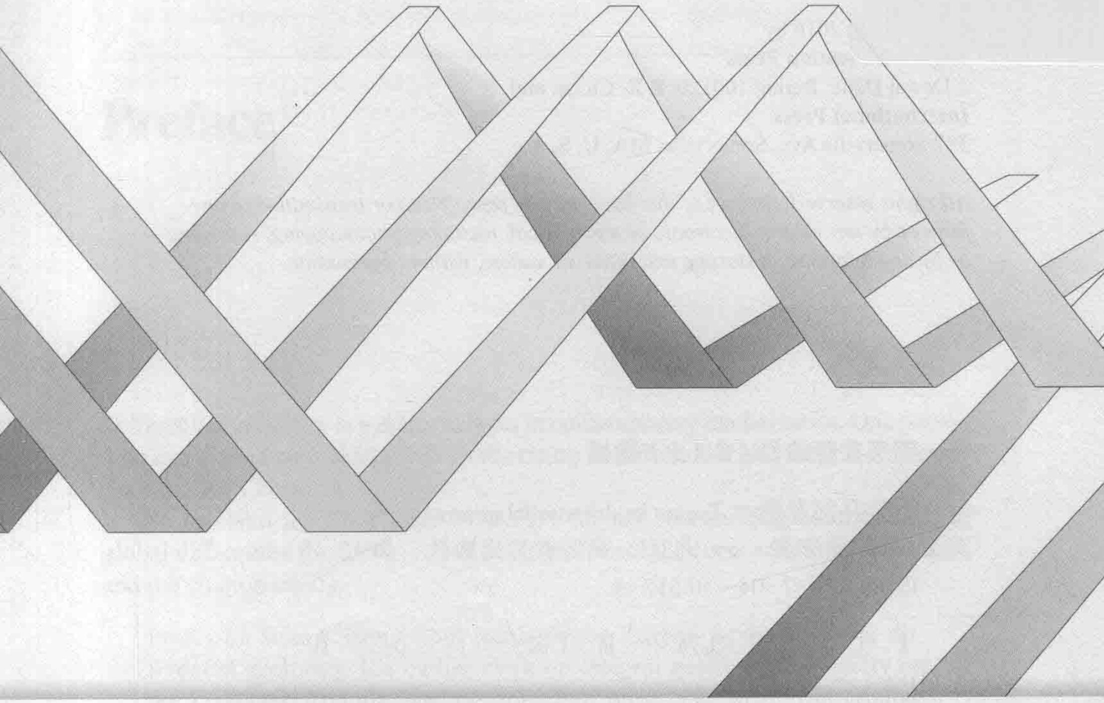
Topics in Differential Geometry

微分几何专题

Shiing-Shen Chern

Higher Education Press

SMM 11
Surveys of Modern Mathematics



Topics in Differential Geometry

微分几何专题

WEIFEN JIHE ZHUANTI

Shiing-Shen Chern

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

International Press

Author
Shiing-Shen Chern

图书在版编目（C I P）数据

微分几何专题 = Topics in differential geometry : 英文 / 陈省身著. -- 北京：高等教育出版社，2016. 10
ISBN 978-7-04-046517-4

Ⅰ. ①微… Ⅱ. ①陈… Ⅲ. ①微分几何 - 英文 Ⅳ. ① O186.1

中国版本图书馆 CIP 数据核字（2016）第 225769 号

策划编辑 王丽萍　　责任编辑 王丽萍　　封面设计 李小璐　　责任印制 毛斯璐

出版发行 高等教育出版社
社　　址 北京市西城区德外大街4号
邮政编码 100120
印　　刷 北京中科印刷有限公司
开　　本 787mm×1092mm 1/16
印　　张 15
字　　数 290 千字
购书热线 010-58581118
咨询电话 400-810-0598
网　　址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
版　　次 2016 年 10 月第 1 版
印　　次 2016 年 10 月第 1 次印刷
定　　价 79.00 元

物 料 号 46517-00

Surveys of Modern Mathematics

Surveys of Modern Mathematics

Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

With joint publication by Higher Education Press (HEP) inside China and International Press (IP) in the West with affordable prices, it is expected that books in this series will broadly reach out to the reader, in particular students, around the world, and hence contribute to mathematics and the world mathematics community.

Series Editors

Preface

Differential geometry is a major subject in contemporary mathematics. One person who had played an essential role in the rising of differential geometry is Professor Shiing-Shen Chern.

He received the Wolf prize in 1983/4 for his "outstanding contributions to global differential geometry, which have profoundly influenced all mathematics", and the citation says:

> Professor Shiing-Shen Chern has been the leading figure in global differential geometry. His earlier work on integral geometry, especially on the kinematic formula, was the source of most later work. His groundbreaking discovery of characteristic classes (now known as Chern classes) was the turning point that set global differential geometry on a course of tumultuous development. The field has blossomed under his leadership, and his results, together with those of his numerous students, have influenced the development of topology, algebraic geometry, complex manifolds, and most recently of gauge theories in mathematical physics.

In 2001, Professor Chern received a special Morningside Lifetime Achievement Award in Mathematics, and the citation reads:

> Professor Chern is awarded the Morningside Lifetime Achievement for his work on developing the foundation of Chinese mathematics, his epochal contributions to research in differential geometry, and his nurturing of leading mathematicians both in China and abroad. In the 1940s, differential geometry was at a low point worldwide; this area of mathematics was only beginning to be understood and to be used. Professor Chern became a pioneer in this subject. Some of his major achievements include the Chern characteristic classes in fiber spaces, and his proof of the Gauss-Bonnet formula. Today, differential geometry is a major subject in mathematics and a large share of the credit for this transformation goes to Professor Chern.

One natural question is how and what did such an initiator of a major subject say about the topics dear to his heart, especially when they were taking shape. What can the young generations, especially the young Chinese mathematicians, learn from a leading figure in the last century? We hope that these two books of lecture notes of Professor Chern and some of his expository papers will answer this question.

Though there are many books now on differential geometry, integral geometry and related topics, they often lack the freshness and directness of the original descriptions by masters. This is consistent with Abel's advice "By studying the masters, not their pupils." This point was endorsed by Professor Chern in his foreword to the Chinese edition of M. Atiyah's collected works:

> No matter how refined or improved a new account is, the original papers on a subject are usually more direct and to the point. When I was young, I was benefited by the advice to read Henri Poincaré, David Hilbert, Felix Klein, Adolf Hurwitz, etc. I did better with Wilhelm Blaschke, Élie Cartan and Heinz Hopf. This has also been in the Chinese tradition, when we were told to read Confucius, Han Yu in prose, and Tu Fu in poetry.

The title of the first book *Topics in Differential Geometry* comes from the title of his lecture notes at IAS, Princeton, in 1951. It also contains his expository papers: "From Triangles to Manifolds", "Curves and Surfaces in Euclidean Space", "Characteristic Classes and Characteristic Forms", "Geometry and Physics", and "The Geometry of G-Structures", together with the set of so far *unpublished lecture notes*: "Minimal Submanifolds in a Riemannian Manifold" .

Together they show how differential geometry is connected to other subjects such as topology and Lie group theory. Though there are more modern expositions of these topics, they are usually not comparable with what Chern wrote. For example, his lecture notes "Topics in Differential Geometry" starts from basics which is accessible to beginners and goes right to some striking applications such as the rigidity theorem of Cohn-Vossen on convex surfaces in $\mathbb{R}^3$ and the rigidity of convex hypersurfaces in $\mathbb{R}^n$. His treatment of the theory of connection has also such characteristic of being direct and going to the key point. The papers selected to be reprinted in this book give an overview of the scope and power of differential geometry. They complement well this set of lecture notes.

This first book will be very valuable to beginners to learn the modern differential geometry, and will also be valuable to experts for them to rethink about differential geometry.

The title of the second book is a combination of titles of two sets of lecture notes of Chern which have not been formally published. It seems that there exists only one known copy of the second set of lecture notes, which is owned by the library of University of Michigan, It also contains a more recent unpublished set of lecture notes titled "Lectures on Differential Geometry".

This second book starts with a paper of Chern which gives a gentle introduction to differential geometry accessible to students and nonexperts, and a survey of the status of global differential geometry in 1971 when Chern gave a plenary talk in ICM. As Gromov commented in his Math Review of this paper:

> This is a brilliant and inspiring exposition. The author begins with a brief historical survey, outlines some fundamental notions and tools, and describes the current situation in four branches of differential geometry: manifolds of positive curvature, curvature and the Euler characteristic, minimal sub manifolds isometric mappings, holomorphic mappings.

This paper also includes some open problems and hence is also very interesting from the historical perspective.

As every student knows, there are two key components of calculus: differentiation and integration. In geometry, there are also two closely related subjects: differential geometry and integral geometry. It is valuable to read and learn them side by side.

The lecture notes "Differential Manifolds" gives a smooth and rapid introduction to differential manifolds and differential geometry. It was delivered by Chern in 1959 at University of Chicago when differential geometry was becoming a major subject in mathematics. Its freshness shines through.

As the citation of the Wolf prize indicates, Chern also made major contribution to integral geometry. The lecture notes "Lectures on Integral Geometry" give an efficient and also accessible introduction to this subject. It is worthwhile to point out that one of Chern's teachers was Blaschke, who was a major or leading figure in integral geometry. This also adds something special to his lectures. In this set of lecture notes, one can also see Chern's global view of mathematics. For example, besides standard topics in integral geometry in Euclidean spaces, it also discusses integral geometry of homogeneous spaces.

This second book will also be a very valuable introduction to both differential and integral geometry for beginners and a supplementary reading for people at all stages.

We hope and believe that there is much one can learn from these collections of lecture notes and papers of Professor Chern, a modern master in mathematics and one of the originators of global differential geometry.

In collecting material for and editing these two books, we have received generous help from Professor Chuu-Lian Terng, and support and blessing from May Chu, the daughter of Processor Chern. Liping Wang of the Higher Education Press has also been very supportive of this project. Without their kind help, these books probably cannot appear in print. We would like thank them sincerely.

Finally, may you enjoy these two books and benefit from them!

Shiu-Yuen Cheng and Lizhen Ji

January 14, 2016.

Contents

1

From Triangles to Manifolds [1]

1.1 Geometry

I believe I am expected to tell you all about geometry; what it is, its developments through the centuries, its current issues and problems, and, if possible, a peep into the future. The first question does not have a clear-cut answer. The meaning of the word *geometry* changes with time and with the speaker. With Euclid, geometry consists of the logical conclusions drawn from a set of axioms. This is clearly not sufficient with the horizons of geometry ever widening. Thus in 1932 the great geometers O. Veblen and J. H. C. Whitehead said, "A branch of mathematics is called geometry, because the name seems good on emotional and traditional grounds to a sufficiently large number of competent people" [1]. This opinion was enthusiastically seconded by the great French geometer Elie Cartan [2]. Being an analyst himself, the great American mathematician George Birkhoff mentioned a "disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis"[3]. Recently my friend Andre Weil said: "The psychological aspects of true geometric intuition will perhaps never be cleared up. At one time it implied primarily the power of visualization in three-dimensional space. Now that higher-dimensional spaces have mostly driven out the more elementary problems, visualization can at best be partial or symbolic. Some degree of tactile imagination seems also to be involved"[4].

At this point it is perhaps better to let things stand and turn to some concrete topics.

1.2 Triangles

Among the simplest geometrical figures is the triangle, which has many beautiful properties. For example, it has one and only one inscribed circle and also one

[1] The American Mathematical Monthly, Vol. 86, No. 5 (May, 1979), pp. 339–349.

and only one circumscribed circle. At the beginning of this century the nine-point circle theorem was known to almost every educated mathematician. But its most intriguing property concerns the sum of its angles. Euclid says that it is equal to 180°, or π by radian measure, and deduces this from a sophisticated axiom, the so-called *parallel axiom*. Efforts to avoid this axiom failed. The result was the discovery of non-Euclidean geometries in which the sum of angles of a triangle is less or greater than π, according as the geometry is hyperbolic or elliptic. The discovery of hyperbolic non-Euclidean geometry, in the eighteenth century by Gauss, John Bolyai, and Lobachevsky, was one of the most brilliant chapters in human intellectual history.

The generalization of a triangle is an n-gon, a polygon with n sides. By cutting the n-gon into $n-2$ triangles, one sees that the sum of its angles is $(n-2)\pi$. It is better to measure the sum of the exterior angles! The latter is equal to 2π, for all n-gons, including triangles.

1.3 Curves in the plane; rotation index and regular homotopy

By applying calculus we can consider smooth curves and closed smooth curves in the plane, i.e., curves with a tangent line everywhere and varying continuously. As a point moves along a closed smooth (oriented) curve C once, the lines through a fixed point O and parallel to the tangent lines of C rotate through an angle $2n\pi$ or rotate n times about O. This integer n is called the rotation index of C (See Fig. 1). A famous theorem in differential geometry says that if C is a simple curve, i.e., if C does not intersect itself $n = \pm 1$.

Clearly, there should be a theorem combining the theorem on the sum of exterior angles of an n-gon and the rotation index theorem of a simple closed smooth curve. This is achieved by considering the wider class of simple closed sectionally smooth curves. The rotation index of such a curve can be defined in a natural way by turning the tangent at a corner an amount equal to the exterior angle(See Fig. 2). Then the rotation index theorem above remains valid for simple closed sectionally smooth curves. In the particular case of an n-gon formed by straight segments, this reduces to the statement that the sum of its exterior angles is 2π.

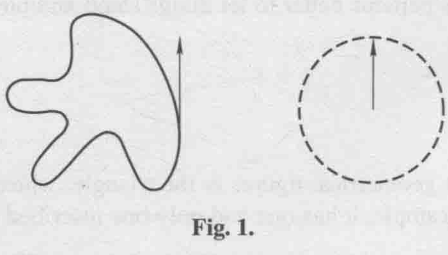


Fig. 1.

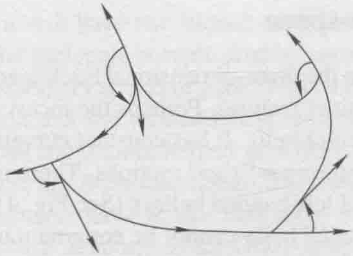


Fig. 2.

This theorem can be further generalized. Instead of simple closed curves we can allow closed curves to intersect themselves. A generic self-intersection can be assigned a sign. Then, if the curve is properly oriented, the rotation index is equal to one plus the algebraic sum of the number of self-intersections (See Fig. 3). For example, the figure 8 has the rotation index zero.

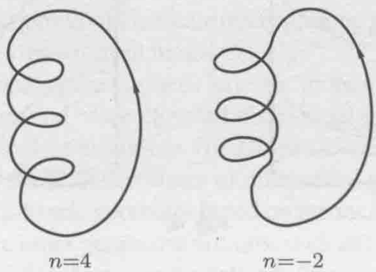


Fig. 3.

A fundamental notion in geometry, or in mathematics in general, is *deformation or homotopy*. Two closed smooth curves are said to be *regularly homotopic* if one can be deformed to the other through a family of closed smooth curves. Since the rotation index is an integer and varies continuously in the family, it must remain a constant; i.e., it keeps the same value when the curve is regularly deformed. A remarkable theorem of Whitney-Graustein says that the converse is true [5]: Two closed smooth curves with the same rotation index are regularly homotopic.

It is a standard practice in mathematics that in order to study closed smooth curves in the plane it is more profitable to look at all curves and to put them into classes, the regular homotopy classes in this case being an example. This may be one of the essential methodological differences between theoretical science and experimental science, where such a procedure is impractical. The Whitney-Graustein theorem says that the only invariant of a regular homotopy class is the rotation index.

1.4 Euclidean three-space

From the plane we pass to the three-dimensional Euclidean space where the geometry is richer and has distinct features. Perhaps the nicest space curve which does not lie in a plane is a circular helix. It has constant curvature and constant torsion and is the only curve admitting ∞^1 rigid motions. There is an essential difference between right-handed and left-handed helices (See Fig. 4), depending on the sign of the torsion; a right-handed helix cannot be congruent to a left-handed one, except by a mirror reflection. Helices play an important role in mechanics. From a geometrical viewpoint it may not be an entire coincidence that the Crick-Watson model of a DNA-molecule is double-helical. A double helix has interesting geometrical properties. In particular, by joining the end points of the helices by segments or arcs, we get two closed curves. In three-dimensional space they have a linking number(See Fig. 5).

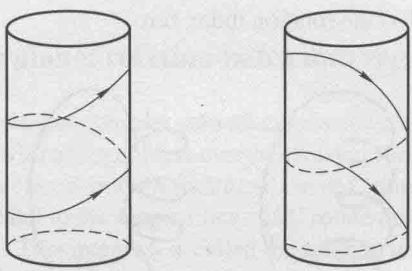


Fig. 4.

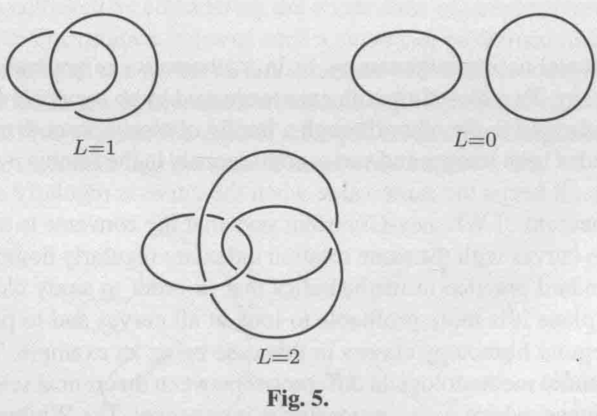


Fig. 5.

A recent controversial issue in biochemistry, raised by the mathematicians William Pohl and George Roberts, is whether the chromosomal DNA is double-helical. In fact, if it is, it will have two closed strands with a linking number of the order of 300,000. The molecule is replicated by separation of the strands and formation of the complementary strand of each. With such a large linking number Pohl and Roberts showed that the replication process would have severe mathematical difficulties. Thus the double-helical structure of the DNA molecule, at least for the chromosome, has been questioned [6]. (Added January 26, 1979; A number of recent experiments have shown that some of the mathematical difficulties for the double helical structure of the DNA-molecule can be overcome by enzymatic activities (cf. F. H. C. Crick, Is DNA really a double helix? preprint, 1978).)

The linking number L is determined by the formula of James H. White [7]:

$$T + W = L, \tag{1}$$

where T is the total twist and W the writhing number. The latter can be experimentally measured and changes by the action of an enzyme. This formula is of fundamental importance in molecular biology. Generally DNA molecules are long. In order to store them in limited space, the most economical way is to writhe and coil them. These discussions could indicate the beginning of a stochastic geometry, with the main examples drawn from biology.

In a three-dimensional space surfaces have far more important properties than curves. Gauss's fundamental work elevated differential geometry from a chapter of calculus to an independent discipline. His *Disquisitiones generales circa superficies curvas* (1827) is the birth certificate of differential geometry. The main idea is that a surface has an intrinsic geometry based on the measure of arc length alone. From the element of arc other geometric notions, such as the angle between curves and the area of a piece of surface, can be defined. Plane geometry is thus generalized to any surface Σ based only on the local properties of the element of arc. This localization of geometry is both original and revolutionary. In place of the straight lines are the geodesics, the "shortest" curves between any two points (sufficiently close). More generally, a curve on Σ has a "geodesic curvature" generalizing the curvature of a plane curve and geodesics are the curves whose geodesic curvature vanishes identically.

Let the surface Σ be smooth and oriented. At every point p of Σ there is a unit normal vector $v(p)$ which is perpendicular to the tangent plane to Σ at p(See Fig. 6). The vector $v(p)$ can be viewed as a point of the unit sphere S_0 with center at the origin of the space. By sending p to $v(p)$ we get the Gauss mapping

$$g : \Sigma \to S_0. \tag{2}$$

The ratio of the element of the area of S_0 by the element of area of Σ under this mapping is called the Gaussian curvature. Gauss's "remarkable theorem" says that the Gaussian curvature depends only on the intrinsic geometry of Σ. In fact, in a sense it characterizes this geometry. Clearly the Gaussian curvature is zero if Σ is the plane.

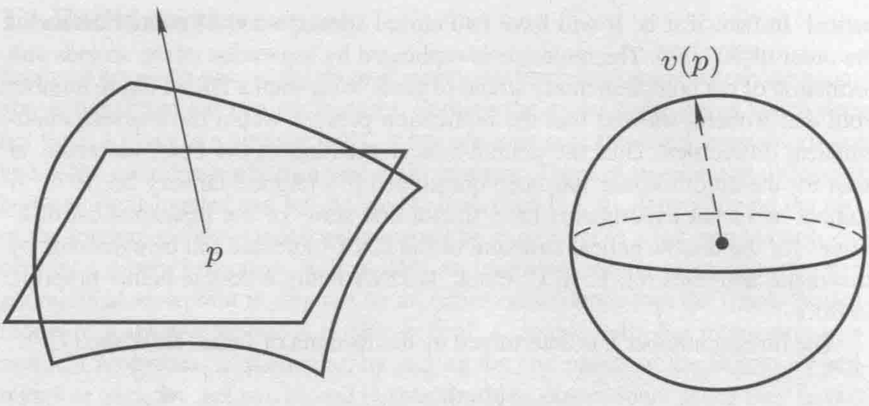


Fig. 6.

As in plane geometry we consider on Σ a domain D bounded by one or more sectionally smooth curves. D has an important topological invariant $\chi(D)$, called its Euler characteristic, which is most easily defined as follows: Cut D into polygons in a "proper way" and denote by v, e, and f the number of vertices, edges, and faces, respectively. Then

$$\chi(D) = v - e + f. \tag{3}$$

(Euler's polyhedral theorem was known before Euler, but Euler seems to have been the first one to recognize explicitly the importance of the "alternating sum".)

The Gauss-Bonnet formula in surface theory is

$$\Sigma \text{ext angles} + \int_{\partial D} \text{geod curv} + \iint_D \text{Gaussian curv} = 2\pi\chi(D), \tag{4}$$

where ∂D is the boundary of D. For a plane domain the Gaussian curvature is zero. If in addition the domain is simply connected, we have $\chi(D) = 1$. Then this formula reduces to the rotation index theorem discussed in §1.3. We are indeed a long way from the sum of angles of a triangle.

Generalizing the geometry of closed plane curves we can consider closed oriented surfaces in space. The generalization of the rotation index is the degree of the Gauss mapping g in (2). The precise definition of the degree is sophisticated. Intuitively it is the number of times that the image $g(\Sigma)$ covers S_0, counted with sign. Unlike the plane, where the rotation index can be any integer, the degree d is completely determined by the topology of Σ; it is equal to

$$d = \frac{1}{2}\chi(\Sigma). \tag{5}$$

For the imbedded unit sphere this degree is $+1$ independently of its orientation. A surprising result of S. Smale [8] says that the two oppositely oriented unit