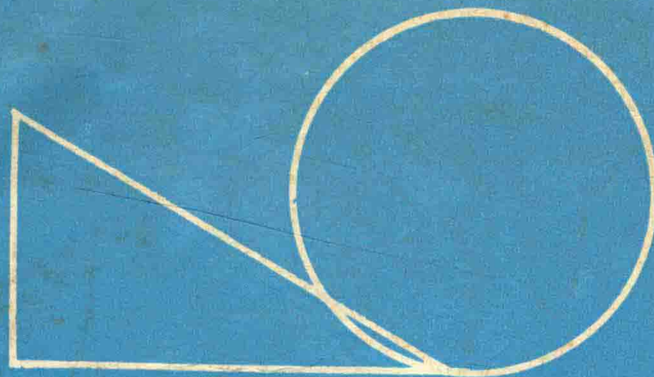

高等数学习题解答

（下册）

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目　　录

第十章　多元函数微分法及其应用

多　元　函　数

10.1 设$x=\frac{1}{2}(1+\sqrt{3})$，$y=\frac{1}{2}(1-\sqrt{3})$，求函数$z=\left[\frac{\operatorname{arctg}(x+y)}{\operatorname{arctg}(x-y)}\right]^2$的值。

解：$z\Big|_{\substack{x=\frac{1}{2}(1+\sqrt{3})\\ y=\frac{1}{2}(1-\sqrt{3})}}=\left(\frac{\operatorname{arcty}1}{\operatorname{arcty}\sqrt{3}}\right)^2=\left(\frac{\frac{\pi}{4}}{\frac{\pi}{3}}\right)^2=\frac{9}{16}$.

10.2 已知函数$f(u、v)=u^v$，试求$f(xy,\ x+y)$。解：$f(xy,\ x+y)=(xy)^{X+Y}$

10.3 已知函数$f(u,\ v,\ w)=u^w+w^{u+v}$，试求：$f(x+y,\ x-y,\ xy)$

解：$f(x+y,\ x-y,\ xy)=(x+y)^{XY}+(xy)^{X+Y+X-Y}=(x+y)^{XY}+(xy)^{2X}$

10.4 试证函数$F(x,\ y)=\ln x\ln y$ 满足关系式：

$F(xy,\ uv)=F(x,\ u)+F(x,\ v)+F(y,\ u)+F(y,\ v)$

证 $F(x,u)+F(x,v)+F(y,u)+F(y,v)$

$=\ln x\ln u+\ln x\ln v+\ln y\ln u+\ln y\ln v$

$=\ln x(\ln u+\ln v)+\ln y(\ln u+\ln v)=\ln(uv)(\ln x+\ln y)$

$=\ln(xy)\cdot\ln(uv)=F(xy,\ uv)$

10.5 试证函数$F(x,\ y)=xy$满足关系式：$F(ax+by,\ cu+dv)$

$=acF(x,\ u)+bcF(y,\ u)+adF(x,\ v)+bdF(y,\ v)$

证 $F(ax+by,\ cu+dv)=(ax+by)(cu+dv)=acxu+bcyu+adxv+bdyv$

$=acF(x,\ u)+bcF(y,\ u)+adF(x,\ v)+bdF(y,\ v)$

10.6 设$F(x,\ y)=\sqrt{x^4+y^4}-2xy$，证明：$F(tx,\ ty)=t^2F(x,\ y)$

证 $F(tx,\ ty)=\sqrt{(tx)^4+(ty)^4}-2(tx)(ty)=t^2\sqrt{x^4+y^4}-t^2(2xy)$

$=t^2(\sqrt{x^4+y^4}-2xy)=t^2F(x,\ y)$

10.7 若函数$z=f(x,\ y)$恒满足关系式$f(tx,\ ty)=t^kf(x,\ y)$，就叫k次齐次函数，试证k次齐次函数$z=f(x,\ y)$能化成$z=x^kF\left(\frac{y}{x}\right)$

证 因为$z=f(x,\ y)$是k次齐次函数，所以有$f(tx,\ ty)=t^kf(x,\ y)$，

令$t=\frac{1}{x}$，有$f\left(1,\ \frac{y}{x}\right)=\frac{1}{x^k}f(x,\ y)$，于是

$f(x,\ y)=x^kf\left(1,\frac{y}{x}\right)=x^kF\left(\frac{y}{x}\right)$

10.8 设$f(x,\ y)=x^2-xy+y^2$，试求：$f(x+\Delta x,\ y)-f(x,\ y)$

及$f(x,\ y+\Delta y)-f(x,\ y)$

若x由2变到2.1，y由2变到1.9，求$f(x+\Delta x,\ y)-f(x,\ y)$

及$f(x, y+\Delta y)-f(x, y)$的值。　　解 $f(x+\Delta x, y)-f(x, y)$

$=(x+\Delta x)^2-(x+\Delta x)y+y^2-(x^2-xy+y^2)$

$=x^2+2\Delta x\cdot x+(\Delta x)^2-xy-\Delta x\cdot y+y^2-x^2+xy-y^2$

$=2\Delta x\cdot x+(\Delta x)^2-\Delta x\cdot y=(2x-y+\Delta x)\Delta x$；$f(x, y+\Delta y)-f(x, y)$

$=x^2-x(y+\Delta y)+(y+\Delta y)^2-(x^2-xy+y)^2$

$=x^2-xy-\Delta y\cdot x+y^2+2\Delta y\cdot y+(\Delta y)^2-x^2+xy-y^2$

$=2\Delta y\cdot y+(\Delta y)^2-\Delta y\cdot x=(2y-x+\Delta y)\Delta y$；

$$f(x+\Delta x, y)-f(x, y)=(2x-y+\Delta x)\Delta x\Big|_{\substack{x=2\\ \Delta x=0.1\\ y=2}}$$

$=2.1\times 0.1=0.21$；

$$f(x, y+\Delta y)-f(x,y)=(2y-x+\Delta y)\Delta y\Big|_{\substack{x=2\\ y=2\\ \Delta y=-0.1}}$$

$=1.9\times(-0.1)=-0.19$

求10.9——10.21题各函数的定义域，并画出其图形：

10.9　$z=x+y$。　解　定义域为xoy平面。

10.10　$z=\frac{4}{x+y}$。解　因$x+y\neq 0$所以定义域为除去直线$y=-x$的xoy平面。

10.11　$z=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$　　解　定义域为xoy平面。

10.12　$z=\ln xy$。　解　因$xy>0$所以定义域为：$x>0$且$y>0$和$x<0$且$y<0$；即不包括坐标轴的第一象限和第三象限部分。

10.13　$z=\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{x-y}}$　　解 因$x+y>0$，$x-y>0$，

所以定义域为：$x+y>0$ 且$x-y>0$

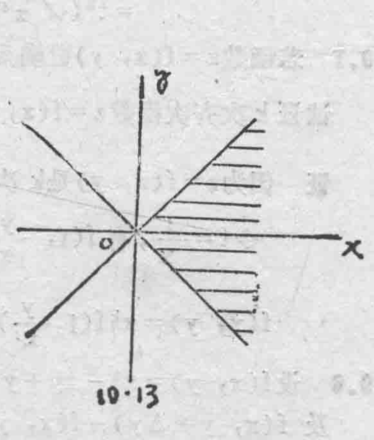
10.14　$z=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

解　$\because 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\geqslant 0$，

$\therefore$定义域为：

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$

即椭圆　$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$及其内部。



10·13

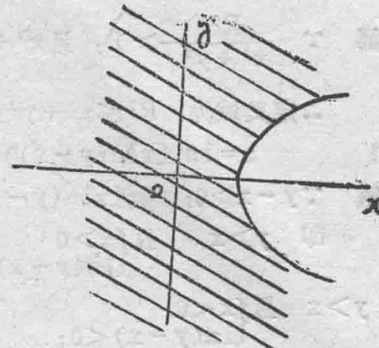
10.15 $z=\ln(y^2-4x+8)$

解 $\because\ y^2-4x+8>0$,

$\therefore$ 定义域为：

$y^2>4x-8$

即 $y^2>4(x-2)$。



10·15

10.16 $z=\arcsin\dfrac{y}{x}$

解 $\because\left|\dfrac{y}{x}\right|\leq 1$, $x\neq 0$

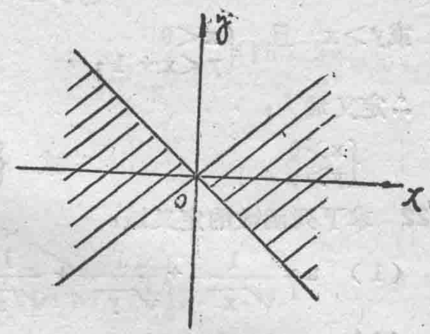
即 $-1\leq\dfrac{y}{x}\leq 1$, $x\neq 0$,

当 $x>0$时，$-x\leq y\leq x$，即

$\begin{cases}y\geqslant -x\\ y\leq x\end{cases}$。

当 $x<0$时，$-x\geqslant y\geqslant x$,

即 $\begin{cases}y\geqslant x\\ y\leq -x\end{cases}$。



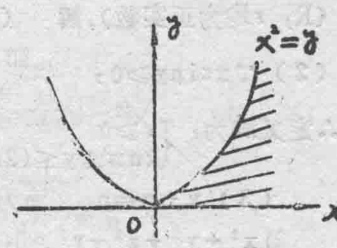
10·16

10.17 $z=\sqrt{x-\sqrt{y}}$

解 $\because y\geqslant 0$，且$x-\sqrt{y}\geqslant 0$,

$\therefore$ 定义域为：

$y\geqslant 0$，$x\geqslant 0$，$x^2\geqslant y$。



10·17

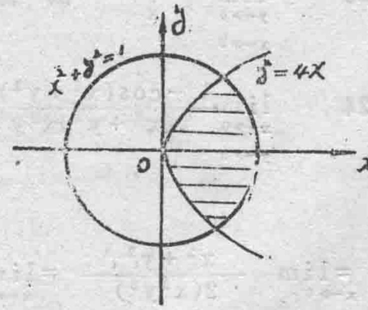
10.18 $z=\dfrac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$

解 $\because 4x-y^2\geqslant 0$且 $1-x^2-y^2>0$。

$1-x^2-y^2\neq 1$,

$\therefore$定义域为：

$4x\geqslant y^2$且$0<x^2+y^2<1$



10·18

10.19 $z=\arcsin\dfrac{x^2+y^2}{4}$

$+\operatorname{arcsec}(x^2+y^2)$

解 $\because\dfrac{x^2+y^2}{4}\leq 1$，且$x^2+y^2\geqslant 1$,

$\therefore$定义域为：$1\leq x^2+y^2\leq 4$,

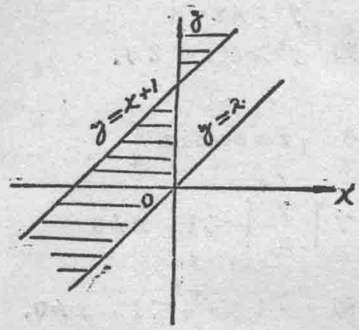
即包括园$x^2+y^2=1$和$x^2+y^2=4$在内的园环域。

10.20 $z=xy+\sqrt{\ln\dfrac{R^2}{x^2+y^2}}+\sqrt{x^2+y^2-R^2}$

解 ∵ $\frac{R^2}{x^2+y^2}\geqslant 1$，且$x^2+y^2\neq 0$，$x^2+y^2-R^2\geqslant 0$，

∴定义域为：$R^2\leq x^2+y^2\leq R^2$ 即 $x^2+y^2=R^2$

10.21 $z=\ln[x\ln(y-x)]$



10·21

解 ∵$y-x>0$， 且$x\ln(y-x)>0$

即 $y>x$ 且$\begin{cases}x>0\\ \ln(y-x)>0,\end{cases}$

或 $y>x$ 且$\begin{cases}x<0\\ \ln(y-x)<0;\end{cases}$

即$y>x$ 且 $\begin{cases}x>0\\ y>x+1,\end{cases}$

或$y>x$ 且 $\begin{cases}x<0\\ y<x+1;\end{cases}$

∴定义域为：

$\begin{cases}x>0\\ y>x+1\end{cases}$ 或 $\begin{cases}x<0\\ x<y<x+1.\end{cases}$

10.22 求下列函数的定义域：

(1) $u=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}$ (2) $z=\sqrt{x\sin y}$

(3) $z=\text{ctg}\pi(x+y)$ (4) $u=\sqrt{R^2-x^2-x^2-z^2}+\frac{1}{\sqrt{x^2+y^2+z^2-r^2}}$

(R，r均为正实数).解 (1) $x>0$ 且$y>0$，$z>0$

(2) ∵$x\sin y\geqslant 0$， 即 $\begin{cases}x\geqslant 0\\ \sin y\geqslant 0\end{cases}$ 或 $\begin{cases}x\leq 0\\ \sin y\leq 0,\end{cases}$

∴定义域为：$\begin{cases}x\geqslant 0\\ 2n\pi\leq y\leq(2n+1)\pi\end{cases}$ 或 $\begin{cases}x\leq 0\\ (2n+1)\pi\leq y\leq(2n+2)\pi\end{cases}$

(3) $x+y\neq n$ n为整数。 (4) ∵$R^2-x^2-y^2-z^2\geqslant 0$

且$x^2+y^2+z^2-r^2>0$， ∴定义域为：$r^2<x^2+y^2+z^2\leq R^2$

求10.23——10.30各题的极限：

10.23 $\lim\limits_{\substack{y\to 0\\ x\to 0}}\frac{\sin xy}{x}$ 解 原式$=\lim\limits_{\substack{x\to 0\\ y\to 0}} y\frac{\sin xy}{xy}=0$

10.24 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$.解：原式$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{2\sin^2\frac{x^2+y^2}{2}(x^2+y^2)}{4\cdot\frac{(x^2+y^2)}{4}\ x^2y^2}$

$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{2(x^2y^2)}=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\frac{1}{y^2}+\frac{1}{x^2}}{2}=\infty$

10.25 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{2-\sqrt{xy+4}}{xy}$.解： 原式$=\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{-xy}{xy(2+\sqrt{xy+4})}=-\frac{1}{4}$

10.26 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}}$. 解：当x，y同号时，

原式 $= \lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{\frac{x^2y^2}{x^2+y^2}} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{\frac{1}{\frac{1}{y^2}+\frac{1}{x^2}}} = 0$，　当x，y反号时，

原式 $= \lim\limits_{\substack{x\to 0\\ y\to 0}} -\sqrt{\frac{x^2y^2}{x^2+y^2}} = 0$，　所以 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = 0$

10.27 $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} (x^2+y^2)e^{-(x+y)}$　解：$\because 0 \le x^2+y^2 \le (x+y)^2$，

又 $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} (x+y)^2 e^{-(x+y)} = \lim\limits_{u\to +\infty} u^2 e^{-u} = \lim\limits_{u\to +\infty} \frac{u^2}{e^u} = \lim\limits_{u\to +\infty} \frac{2u}{e^u} = 0$，

$\therefore \lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} (x^2+y^2)e^{-(x+y)} = 0$

10.28 $\lim\limits_{\substack{x\to 0\\ y\to 0}} (1+xy)^{\frac{1}{x}}$. 解：原式 $= \lim\limits_{\substack{x\to 0\\ y\to 0}} \left[(1+xy)^{\frac{1}{xy}}\right]^y = e^0 = 1$

10.29 $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2}$. 解：$\because 2xy \le x^2+y^2$，

$\therefore\quad 0 \le \left(\frac{xy}{x^2+y^2}\right)^{x^2} \le \left(\frac{xy}{2xy}\right)^{x^2} = \left(\frac{1}{2}\right)^{x^2}$，

而 $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} \left(\frac{1}{2}\right)^{x^2} = 0$，　$\therefore \lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0$

10.30 $\lim\limits_{\substack{x\to 1\\ y\to 0}} \frac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$. 解：　原式 $= \frac{\ln 2}{\sqrt{1}} = \ln 2$

10.31 证明：当$x\to 0$，$y\to 0$时，函数 $u = \frac{y}{x-y}$ 的极限不存在，(x，y)以怎样的方式趋近于(0，0)时，能使：$\lim u = 3$，$\lim u = 2$，$\lim u = -2$.

证 $\because \lim\limits_{\substack{x\to 0\\ y\to 0\\ (x=0)}} \frac{y}{x-y} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{y}{-y} = -1$，$\lim\limits_{\substack{x\to 0\\ y\to 0\\ (y=0)}} \frac{y}{x-y} = 0$，

$\therefore$ $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{y}{x-y}$ 不存在，又取 $y = kx$，有

$$\lim_{\substack{x\to 0\\ y\to 0\\ (y=kx)}} \frac{y}{x-y}=\lim_{\substack{x\to 0\\ y\to 0}} \frac{kx}{x-kx}=\frac{k}{1-k}.$$

1。要 $\lim u=\frac{k}{1-k}=3$， $k=\frac{3}{4}$ 即(x, y)沿 $y=\frac{3}{4}x$ 趋近$(0, 0)$

2° 要 $\lim u=2$， $k=\frac{2}{3}$ 即(x, y)沿 $y=\frac{2}{3}x$ 趋近于$(0, 0)$

3° 要 $\lim u=-2$， $k=2$ 即(x, y)沿 $y=2x$ 趋近于$(0, 0)$

10.32 下列函数当$(x, y)\to(0, 0)$时，极限是否存在？

（1）$f(x, y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$　　（2）$f(x,y)=\frac{x^2y}{x^4+y^2}$

解：（1）取 $y=kx$，则 $\lim\limits_{\substack{x\to 0\\ y\to 0\\ (y=kx)}} \frac{x^2y^2}{x^2y^2+(x-y)^2}$

$$=\lim_{\substack{x\to 0\\ y\to 0}} \frac{k^2x^4}{k^2x^4+(x-kx)^2}=\lim_{\substack{x\to 0\\ y\to 0}} \frac{k^2x^2}{k^2x^2+(1-k)^2},$$

当 $k=1$ 时， $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x\ y)=1$； 当 $k\neq 1$ 时 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x\ y)=0$；

故 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x\ y)$ 不存在。　　（2）当 $y=x$ 时。

$$\lim_{\substack{x\to 0\\ y\to 0\\ (y=x)}} \frac{x^2y}{x^4+y^2}=\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3}{x^4+x^2}=\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{x^2+1}=0;$$

当 $y=x^2$ 时 $\lim\limits_{\substack{x\to 0\\ y\to 0\\ (y=x2)}} \frac{x^2y}{x^4+y^2}=\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^4}{x^4+x^4}=\frac{1}{2}$；故 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x\ y)$ 不存在。

10.33 下列函数在何处间断：（1） $z=\frac{y^2+2x}{y^2-2x}$　　（2） $z=\ln|x-y|$

解：（1）由于 $y^2-2x=0$ 时函数没有意义，故函数在抛物线 $y^2=2x$ 上间断。

（2）由于 $x-y=0$ 时函数没有意义，故函数在直线 $y=x$ 上间断。

10.34 证明函数 $z=x^2+y^2$ 在全平面连续。 **证**：设(x_0, y_0)为平面任一点，

$\because |x^2+y^2-(x_0^2+y_0^2)|=|(x^2-x_0^2)+(y^2-y_0^2)|$

$\leq|x^2-x_0^2|+|y^2-y_0^2|=|x-x_0||x+x_0|+|y-y_0||y+y_0|$

$\leq|x-x_0|(2|x_0|+1)+|y-y_0|(2|y_0|+1)$　　（$\because$ $x\to x_0$，

设 $|x-x_0|<1$， $|x|-x_0|\leq|x-x_0|<1$， $|x|<|x_0|+1$，

$\therefore |x+x_0|\leq|x|+|x_0|<2|x_0|+1$)，对于 $\varepsilon>0$，要使

$|x^2+y^2-(x_0^2+y_0^2)|<\varepsilon$，只需 $|x-x_0|<\frac{\varepsilon}{2(2|x_0|+1)}$，

$|y-y_0|<\frac{\varepsilon}{2(2|y_0|+)}$,

因此，存在 $\delta=\min\left[\frac{\varepsilon}{2(2|x_0|+1)},\frac{\varepsilon}{2(2|y_0|+1)}\right]$，当 $|x-x_0|<\delta$，

$|y-y_0|<\delta$，就有 $|x^2+y^2-(x_0^2+y_0^2)|<\varepsilon$，由于 (x_0,y_0) 的任意性，所以函数 x^2+y^2 在整个平面连续。

10.35 设 $f(x,y)$ 在区域D内一点 $P(a,b)$ 是连续的，且 $f(a,b)>0$，证明：存在点P的一个邻域，使 $f(x,y)>0$。证 ∵ $f(x,y)$ 在 $P(a,b)$ 处连续，

∴对 $\frac{1}{2}f(a,b)>\varepsilon>0$，存在 $\delta>0$，当 $\sqrt{(x-a)^2+(y-b)^2}<\delta$ 时，有

$|f(x,y)-f(a,b)|<\varepsilon$ 即 $f(x,y)>f(a,b)-\varepsilon>0$。

一 阶 导 数

10.36 证明：若 $f(x,y)$ 的偏导数存在，则有 $f_x'(x,b)=\frac{d}{dx}f(x,b)$

证： 因 $f(x,y)$ 的偏导数存在，按定义，有 $f_x'(x,b)=f_x'(x,y)|_{y=b}$

$$=\lim_{\Delta x\to 0}\left.\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\right|_{y=b}$$

$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,b)-f(x,b)}{\Delta x}$$

当 $y=b$ 时，$f(x,y)$ 成为一元函数 $f(x,b)$，其导数 $\frac{d}{dx}f(x,b)$

正是上述极限，所以 $f_x'(x,b)=\frac{d}{dx}f(x,b)$

10.37 设 $f(x,y)=x+y-\sqrt{x^2+y^2}$，求 $f_x'(3,4)$

解： ∵ $f_x'(x,y)=1-\frac{x}{\sqrt{x^2+y^2}}$

∴ $f_x'(3,4)=1-\frac{3}{\sqrt{3^2+4^2}}=1-\frac{3}{5}=\frac{2}{5}$

10.38 设 $z=\ln(x+\frac{y}{2x})$，求 $\left.\frac{\partial z}{\partial z}\right|_{\substack{x=1\\y=0}}$

解：∵ $\frac{\partial z}{\partial y}=\frac{1}{x+\frac{y}{2x}}\cdot\frac{1}{2x}=\frac{1}{2x^2+y}$

∴ $\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}}=\left.\frac{1}{2x^2+y}\right|_{\substack{x=1\\y=0}}=\frac{1}{2}$

10.39 设 $z=(1+xy)^y$，求 $\left[\frac{\partial z}{\partial x}\right]_{\substack{x=1\\y=1}}$ 及 $\left[\frac{\partial z}{\partial y}\right]_{\substack{x=1\\y=1}}$

解：∵ $\frac{\partial z}{\partial x}=y(1+xy)^{y-1}y=y^2(1+xy)^{y-1}$

$\frac{\partial z}{\partial y}=(1+xy)^y\left[\frac{xy}{1+xy}+\ln(1+xy)\right]$

∴ $\left[\frac{\partial z}{\partial x}\right]_{\substack{x=1\\y=1}}=y^2(1+xy)^{y-1}\Big|_{\substack{x=1\\y=1}}=1$

$\left[\frac{\partial z}{\partial y}\right]_{\substack{x=1\\y=1}}=2\times\left[\frac{1}{2}+\ln2\right]=1+2\ln2$

10.40 $z=\frac{x\cos y-y\cos x}{1+\sin x+\sin y}$，求 $\frac{\partial z}{\partial x}\Big|_{\substack{x=0\\y=0}}$，$\frac{\partial z}{\partial y}\Big|_{\substack{x=0\\y=0}}$

解：$\frac{\partial z}{\partial x}=\frac{(\cos y+y\sin x)(1+\sin x+\sin y)-(x\cos y-y\cos x)\cos x}{(1+\sin x+\sin y)^2}$

$\frac{\partial z}{\partial y}=\frac{(-x\sin y-\cos x)(1+\sin x+\sin y)-(x\cos y-y\cos x)(\cos y)}{(1+\sin x+\sin y)^2}$

∴ $\frac{\partial z}{\partial x}\Big|_{\substack{x=0\\y=0}}=\frac{1-0}{1}=1$，$\frac{\partial z}{\partial y}\Big|_{\substack{x=0\\y=0}}=\frac{-1-0}{1}=-1$

10.41 设 $u=\sqrt{\sin^2x+\sin^2y+\sin^2z}$，求 $\frac{\partial u}{\partial z}\Big|_{\substack{y=0\\x=0\\z=\frac{\pi}{4}}}$

解：$\frac{\partial z}{\partial u}=\sqrt{\frac{\sin z\cos z}{\sin^2x+\sin^2y+\sin^2z}}$，$\frac{eu}{ez}\Big|_{\substack{x=0\\y=0\\z=\frac{\pi}{4}}}=\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{2}$

10.42 设 $f(x,y)=e^{-x}\sin(x+2y)$，求 $f_x'\left(0,\frac{\pi}{4}\right)$、$f_y'\left(0,\frac{\pi}{4}\right)$

解：∵ $f_x'(x\ \ y)=-e^{-x}\sin(x+2y)+e^{-x}\cos(x+2y)$

$f_y'(x,y)=2e^{-x}\cos(x+2y)$

∴ $f_x'(0,\frac{\pi}{4})=-\sin\frac{\pi}{2}+\cos\frac{\pi}{2}=-1$，$f_y'(0,\frac{\pi}{4})=2\times1\times0=0$

10.43 设 $u=\ln(1+x+y^2+z^3)$，当 $x=y=y=1$ 时，求 $u_x'+u_y'+u_z'$

解：∵ $u_x'=\frac{1}{1+x+y^2+z^3}$，$u_y'=\frac{2y}{1+x+y^2+z^3}$，

$u_z'=\frac{3y^2}{1+x+y^2+z^3}$，∴ $u_x'+u_y'+u_z'=\frac{1+2y+3z^2}{1+x+y^2+z^3}$，

$u_z'+u_y'+{}_z'\Big|_{x=y=z=1}=\frac{6}{4}=\frac{3}{2}$

求 **10.44——10.62** 各函数对各自变量的偏导数：

10.44 $z=\frac{y}{x}$ 解：$\frac{\partial z}{\partial x}=-\frac{y}{x^2}$，$\frac{\partial z}{\partial y}=\frac{1}{x}$

10.45 $z=\text{arctg}\frac{y}{x}$ 解：$\frac{\partial z}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

10.46 $z=\ln\left(\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt[3]{y}}\right)$. 解：$\frac{\partial z}{\partial x}=\frac{1}{\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt[3]{y}}}\cdot\left(-\frac{1}{3}x^{-\frac{4}{3}}\right)$

$$=\frac{-\frac{1}{3}x^{-\frac{4}{3}}\left(\sqrt[3]{x}\sqrt[3]{y}\right)}{\sqrt[3]{y}-\sqrt[3]{x}}=\frac{-\sqrt[3]{y}}{3x(\sqrt[3]{y}-\sqrt[3]{x})}$$

$$=\frac{\sqrt[3]{y}}{3x\left(\sqrt[3]{x}-\sqrt[3]{y}\right)},\quad \frac{\partial z}{\partial y}=\frac{1}{\frac{1}{\sqrt[3]{x}}-\frac{1}{\sqrt[3]{y}}}\left(\frac{1}{3}y^{-\frac{4}{3}}\right)$$

$$=\frac{\sqrt[3]{x}}{3y\left(\sqrt[3]{y}-\sqrt[3]{x}\right)}$$

10.47 $u=\sqrt{x^2+y^2-2xy\cos z}$

解：$\frac{\partial u}{\partial x}=\frac{2x-2y\cos z}{2\sqrt{x^2+y^2-2xy\cos z}}=\frac{x-y\cos z}{\sqrt{x^2+y^2-2xy\cos z}}$

$$\frac{\partial u}{\partial y}=\frac{y-x\cos z}{\sqrt{x^2+y^2-2xy\cos z}},\quad \frac{\partial u}{\partial z}=\frac{2xy\sin z}{2\sqrt{x^2+y^2-2xy\cos z}}$$

$$=\frac{xy\sin z}{\sqrt{x^2+y^2-2xy\cos z}}$$

10.48 $z=(\sin x)^{\cos y}$ 解：$\frac{\partial z}{\partial x}=\cos y(\sin x)^{\cos y-1}\cdot\cos x$

$$=\cos y\,\text{ctg}x(\sin x)^{\cos y},\quad \frac{\partial z}{\partial y}=(\sin x)^{\cos y}\ln\sin x\cdot(-\sin x)$$

$$=-\sin x\ln\sin x(\sin x)^{\cos y}$$

10.49 $u=\frac{2x-t}{x+2t}$ 解：$\frac{\partial u}{\partial x}=\frac{2(x+2t)-(2x-t)}{(x+2t)^2}=\frac{5t}{(x+2t)^2}$

$$\frac{\partial u}{\partial t}=\frac{-(x+2t)-2(2x-t)}{(x+2t)^2}=\frac{-5x}{(x+2t)^2}$$

10.50 $u=\arcsin(t\sqrt{x})$ 解：$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-t^2x}}\cdot\frac{t}{2\sqrt{x}}$

$=\frac{t}{2\sqrt{x-t^2x^2}}$，$\frac{\partial u}{\partial t}=\frac{\sqrt{x}}{\sqrt{1-t^2x}}=\sqrt{\frac{x}{1-t^2x}}$

10.51 $z=\frac{xe^y}{y^2}$ 解：$\frac{\partial z}{\partial x}=\frac{e^y}{y^2}$，$\frac{\partial z}{\partial y}=\frac{xe^yy^2-2xe^yy}{y^4}$

$=\frac{x(y-2)e^y}{y^3}$

10.52 $z=\ln\operatorname{tg}\frac{x}{y}$ 解：$\frac{\partial z}{\partial x}=\frac{1}{\operatorname{tg}\frac{x}{y}}\sec^2\frac{x}{y}\cdot\frac{1}{y}$

$=\frac{2}{y\sin\frac{2x}{y}}$，$\frac{\partial z}{\partial y}=\frac{1}{\operatorname{tg}\frac{x}{y}}\sec^2\frac{x}{y}\left(-\frac{x}{y^2}\right)=\frac{-2x}{y^2\sin\frac{2x}{y}}$

10.53 $z=\ln\sin(x-2y)$ 解：$\frac{\partial z}{\partial x}=\frac{\cos(x-2y)}{\sin(x-2y)}=\operatorname{ctg}(x-2y)$，

$\frac{\partial z}{\partial y}=\frac{-2\cos(x-2y)}{\sin(x-2y)}=-2\operatorname{ctg}(x-2y)$，

10.54 $z=\left(\frac{1}{3}\right)^{-\frac{y}{x}}$ 解：$\frac{\partial z}{\partial x}=\left(\frac{1}{3}\right)^{-\frac{y}{x}}\ln\frac{1}{3}\cdot\frac{y}{x^2}$

$=-\frac{y}{x^2}\left(\frac{1}{3}\right)^{-\frac{y}{x}}\ln 3$，$\frac{\partial z}{\partial y}=\left(\frac{1}{3}\right)^{-\frac{x}{y}}\ln\frac{1}{3}\cdot\left(-\frac{1}{x}\right)$

$=\frac{1}{x}\left(\frac{1}{3}\right)^{\frac{-y}{x}}\ln 3$

10.55 $l=\rho e^{\pi\cos\varphi}$ 解：$\frac{\partial l}{\partial\rho}=e^{\pi\cos\varphi}$

$\frac{\partial l}{\partial\varphi}=\rho e^{\pi\cos\varphi}(-\pi\sin\varphi)=-\pi\rho\sin\varphi e^{\pi\cos\varphi}$

10.56 $z=xye^{\sin\pi xy}$ 解：$\frac{\partial z}{\partial x}=ye^{\sin\pi xy}+xye^{\sin\pi xy}\pi y\cos\pi xy$

$=ye^{\sin\pi xy}(1+\pi xy\cos\pi xy)$，$\frac{\partial z}{\partial y}=xe^{\sin\pi xy}+xye^{\sin\pi xy}\cdot\pi x\cos\pi xy$

$=xe^{\sin\pi xy}(1+\pi xy\cos\pi xy)$

10.57 $z=\ln(x+\ln y)$ 解：$\frac{\partial z}{\partial x}=\frac{1}{x+\ln y}$

$\frac{\partial z}{\partial y}=\frac{1}{x+\ln y}\cdot\frac{1}{y}=\frac{1}{y(x+\ln y)}$

10.58 $z=\sqrt{x}\sin\frac{y}{x}$ 解：$\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x}}\sin\frac{y}{x}+\sqrt{x}\cos\frac{y}{x}\cdot(-\frac{y}{x^2})$

$=\frac{1}{2\sqrt{x}}\sin\frac{y}{x}-\frac{y}{x^2}\sqrt{x}\cos\frac{y}{x}$，$\frac{\partial z}{\partial y}=\sqrt{x}\cos\frac{y}{x}\cdot\frac{1}{x}=\frac{1}{\sqrt{x}}\cos\frac{y}{x}$

10.59 $u=x^{\frac{y}{z}}$ 解：$\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}=\frac{y}{xz}x^{\frac{y}{z}}$，

$\frac{\partial u}{\partial y}=x^{\frac{y}{z}}\ln x\cdot\frac{1}{z}=\frac{1}{z}x^{\frac{y}{z}}\ln x$，$\frac{\partial u}{\partial z}=x^{\frac{y}{z}}\ln x(-\frac{y}{z^2})$

$=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$

10.60 $u=x^{y^z}$ 解：$\frac{\partial u}{\partial x}=y^z x^{y^z-1}=\frac{y^z}{x}\cdot x^{y^z}$，

$\frac{\partial u}{\partial y}=x^{y^z}\ln x\cdot zy^{z-1}=\frac{zy^z\ln x}{y}\cdot x^{y^z}$，

$\frac{\partial u}{\partial z}=x^{y^z}\ln x\cdot y^z\ln y=y^z x^{y^z}\ln x\cdot\ln y$

10.61 $u=\rho e^{t\varphi}+e^{-\varphi}+t$ 解 $\frac{\partial u}{\partial\rho}=e^{t\varphi}$，$\frac{\partial u}{\partial\varphi}=\rho e^{t\varphi}\cdot t-e^{-\varphi}=\rho te^{t\varphi}-e^{-\varphi}$，

$\frac{\partial u}{\partial t}=\rho e^{t\varphi}\cdot\varphi+1=\rho\varphi e^{t\varphi}+1$

10.62 $u=e^{\varphi+\theta}\cos(\theta-\varphi)$ 解：$\frac{\partial u}{\partial\varphi}=e^{\varphi+\theta}\cos(\theta-\varphi)+e^{\varphi+\theta}\sin(\theta-\varphi)$

$=e^{\varphi+\theta}[\cos(\theta-\varphi)+\sin(\theta-\varphi)]$，$\frac{\partial u}{\partial\varphi}=e^{\varphi+\theta}\cos(\theta-\varphi)-e^{\varphi+\theta}\sin(\theta-\varphi)$

$=e^{\varphi+\theta}\cos(\theta-\varphi)-\sin(\theta-\varphi)$

10.63 求曲线 $\begin{cases}z=\frac{x^2+y^2}{4}\\ y=4\end{cases}$ 在点（2，4，5）处的切线与横轴正向所成的角度。

解：因 $\frac{\partial z}{\partial x}=\frac{2x}{4}=\frac{x}{2}$，故 $\left.\frac{\partial z}{\partial x}\right|_{(2,4,5)}=1$

即

$\operatorname{tg}(\widehat{\vec{T}\ \vec{i}})=1$，$(\widehat{\vec{T},\ \vec{i}})=\frac{\pi}{4}$

10.64 求曲线：$\begin{cases}z=\sqrt{1+x^2+y^2}\\ x=1\end{cases}$ 在点$(1,1,\sqrt{3})$处的切线与y轴正向所成的角度。解：$\because\frac{\partial z}{\partial y}=\frac{y}{\sqrt{1+x^2+y^2}}$，$\therefore\left.\frac{\partial z}{\partial y}\right|_{(1,1\sqrt{3})}=\frac{1}{\sqrt{3}}$

即 $\operatorname{tg}(\widehat{\vec{T},\ \vec{j}})=\frac{1}{\sqrt{3}}$，$(\widehat{\vec{T},\ \vec{j}})=\frac{\pi}{6}$

10.65 两个曲面 $z = x^2 + \frac{y^2}{6}$ 和 $z = \frac{x^2 + y^2}{3}$ 与平面 $y = 2$ 相交的曲线成什么角度。

解：由方程组 $\begin{cases} z = x^2 + \frac{y^2}{6} \\ z = \frac{x^2 + y^2}{3} \\ y = 2 \end{cases}$ 解得两曲线的交点为 $M\left(\pm 1, 2, \frac{5}{3}\right)$，

又 $\frac{\partial z_1}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + \frac{y^2}{6}\right) = 2x, k_1 = \frac{\partial z_1}{\partial x}\Big|_M = \pm 2$；

$\frac{\partial z_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{3}\right) = \frac{2}{3}x$，$k_2 = \frac{\partial z_2}{\partial x}\Big|_M = \pm\frac{2}{3}$，

$$\therefore \quad tg\theta = \left|\frac{k_1 - k_2}{1 + k_1 k_2}\right| = \left|\frac{\pm 2 \mp \frac{2}{3}}{1 + (\pm 2)\left(\frac{\pm 2}{3}\right)}\right| = \left|\pm\frac{\frac{4}{3}}{1 + \frac{4}{3}}\right| = \frac{4}{7}$$

夹角 $\theta = arctg\frac{4}{7}$

10.66 设 $z = \ln(\sqrt{x} + \sqrt{y})$，证明 $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}$

证：∵ $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x} + \sqrt{y}} \cdot \frac{1}{2\sqrt{x}}$，$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{x} + \sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\sqrt{x}}{2(\sqrt{x} + \sqrt{y})} + \frac{\sqrt{y}}{2(\sqrt{x} + \sqrt{y})} = \frac{1}{2}$$

10.67 设 $z = e^{\frac{x}{y^2}}$，证明 $2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

证：∵ $\frac{\partial z}{\partial x} = \frac{1}{y^2} e^{\frac{x}{y^2}}$ $\qquad \frac{\partial z}{\partial y} = -\frac{2x}{y^3} e^{\frac{x}{y^2}}$

$$\therefore \quad 2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x}{y^2}e^{\frac{x}{y^2}} - \frac{2x}{y^2} e^{\frac{x}{y^2}} = 0$$

10.68 设 $z = x^y$，证明 $\frac{x}{y} \cdot \frac{\partial z}{\partial x} + \frac{1}{\ln x} \frac{\partial z}{\partial y} = 2z$

证：∵ $\frac{\partial z}{\partial x} = yx^{y-1} = \frac{y}{x}x^y$，$\frac{\partial z}{\partial y} = x^y \ln x$

$$\therefore \quad \frac{x}{y} \frac{\partial z}{\partial x} + \frac{1}{\ln x} \frac{\partial z}{\partial y} = x^y + x^y = 2x^y = 2z$$

10.69 设 $T = \pi\sqrt{\frac{l}{g}}$，证明 $l\frac{\partial T}{\partial l} + g\frac{\partial T}{\partial g} = 0$ **证：**∴ $\frac{\partial T}{\partial l} = \frac{\pi}{2\sqrt{lg}}$，

$\frac{\partial T}{\partial g} = -\frac{\sqrt{l}}{2g\sqrt{g}}$，∴ $l\frac{\partial T}{\partial l} + g\frac{\partial T}{\partial g} = \frac{\sqrt{l}}{2\sqrt{g}} - \frac{\sqrt{l}}{2\sqrt{g}} = 0$

10.70 设 $z=\sqrt{x}\sin\frac{y}{x}$， 证明 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{z}{2}$

证：∵ $\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x}}\sin\frac{y}{x}+\sqrt{x}\cos\frac{y}{x}\left(-\frac{y}{x^2}\right)$，

$\frac{\partial z}{\partial y}=\sqrt{x}\cos\frac{y}{x}\cdot\frac{1}{x}=\frac{\cos\frac{y}{x}}{\sqrt{x}}$ ∴ $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

$=\frac{\sqrt{x}}{2}\sin\frac{y}{x}-\frac{y}{\sqrt{x}}\cos\frac{y}{x}+\frac{y}{\sqrt{x}}\cos\frac{y}{x}=\frac{\sqrt{x}}{2}\sin\frac{y}{x}=\frac{z}{2}$

10.71 设 $u=\sqrt{x^2+y^2+z^2}$，证明

$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$ 证：∵ $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$，

$\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}$， $\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$，

∴ $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1$

10.72 设 $z=e^{\frac{x}{y}}\ln y$， 证明 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{z}{\ln y}$

证：∵ $\frac{\partial z}{\partial x}=e^{\frac{x}{y}}\cdot\frac{1}{y}\ln y$， $\frac{\partial z}{\partial y}=e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)\ln y+\frac{1}{y}e^{\frac{x}{y}}$，

∴ $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x}{y}e^{\frac{x}{y}}\ln y-\frac{x}{y}e^{\frac{x}{y}}\ln y+e^{\frac{x}{y}}=e^{\frac{x}{y}}=\frac{z}{\ln y}$

全微分及其应用

求**10.73——10.77**题各函数的偏导数和全微分：

10.73 $z=\frac{xy}{x-y}$ **解：** $\frac{\partial z}{\partial x}=\frac{y(x-y)-xy}{(x-y)^2}=\frac{-y^2}{(x-y)^2}$，

$\frac{\partial z}{\partial y}=\frac{x(x-y)+xy}{(x-y)^2}=\frac{x^2}{(x-y)^2}$，

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{-y^2dx+x^2dy}{(x-y)^2}$

10.74 $z=\arcsin\frac{x}{y}$ **解：** $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y}=\frac{1}{\sqrt{y^2-x^2}}$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right)=-\frac{x}{y\sqrt{y^2-x^2}}$$

$$dz=\frac{1}{\sqrt{y^2-x^2}}dx-\frac{x}{y\sqrt{y^2-x^2}}dy=\frac{ydx-xdy}{y\sqrt{y^2-x^2}}$$

10.75 $z=\sin xy$ 解：$\frac{\partial z}{\partial x}=y\cos xy$，$\frac{\partial z}{\partial y}=x\cos xy$，

$$dz=y\cos xydx+x\cos xydy=\cos xy(ydx+xdy)$$

10.76 $u=\frac{y}{x}+\frac{z}{y}-\frac{x}{z}$ 解：$\frac{\partial u}{\partial x}=-\frac{y}{x^2}-\frac{1}{z}$

$$\frac{\partial u}{\partial y}=\frac{1}{x}-\frac{z}{y^2}\qquad \frac{\partial u}{\partial z}=\frac{1}{y}+\frac{x}{z^2}$$

$$dz=\left(-\frac{y}{x^2}-\frac{1}{z}\right)dx+\left(\frac{1}{x}-\frac{z}{y^2}\right)dy+\left(\frac{1}{y}+\frac{x}{z^2}\right)dz$$

10.77 $u=e^{x(x^2+y^2+z^2)}$ 解 $\frac{\partial u}{\partial x}=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$

$$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)}\qquad \frac{\partial u}{\partial z}=2xze^{x(x^2+y^2+z^2)}$$

$$du=e^{x(x^2+y^2+z^2)}[(3x^2+y^2+z^2)dx+2xydy+2xzdz]$$

求**10.78——10.81**题各函数的全微分：

10.78 $z=x^2y$ 解：$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=2xydx+x^2dy$

10.79 $u=\frac{s+t}{s-t}$ 解：∵ $\frac{\partial u}{\partial s}=\frac{(s-t)-(s+t)}{(s-t)^2}=-\frac{2t}{(s-t)^2}$

$$\frac{\partial u}{\partial t}=\frac{(s-t)+(s+t)}{(s-t)^2}=\frac{2s}{(s-t)^2}$$

$$du=-\frac{2t}{(s-t)^2}ds+\frac{2s}{(s-t)^2}dt=\frac{2(sdt-tds)}{(s-t)^2}$$

10.80 $z=x\ln y$

解：$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\ln ydx+\frac{x}{y}dy$

10.81 $u=\ln(3x-2y+z)$

解：$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$$=\frac{3}{3x-2y+z}dx+\frac{-2}{3x-2y+z}dy+\frac{1}{3x-2y+z}dz$$

$$=\frac{3dx-2dy+dz}{3x-2y+z}$$