

山东省精品课程教材

山东省双语示范课程教材

FUNCTIONS OF COMPLEX VARIABLES


复变函数论

第二版

Ma Lixin

马立新 (编)

CHINA AGRICULTURE PRESS

 中国农业出版社

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前 言



[复 变 函 数 论]

伴随着我国高等教育改革形势的发展,高等教育的人才培养模式和教学方式以及教学方法正在发生重大变化。教育部于2001年在《关于加强高等学校本科教学工作提高教学质量的若干意见》(教高【2001】4号)中,明确要求高校要积极开展公共课和专业课教学双语教学的研究和实践。自2005年开始,德州学院采取一系列措施,开展这项创新教育改革活动,推出第一批英汉双语专业课程教学,《复变函数论》即是其中之一。对学生调查的结果及学校组织的听课评议结果均反映良好,其特色得到了山东省课程建设委员会专家的充分肯定,2009年被评为山东省首批双语示范课程、2012年被评为省级精品课程。

对于双语教学,选一本合适的教材极其重要。由于教育体制不同,我国高校复变函数论教材与英美原版教材差别较大,很难找到完全适合我国开展《复变函数论》双语教学的英文原版教材。自2006年起,德州学院选用了James Ward Brown & Ruel V. Churchill编写的英文原版教材Complex Variables and Applications(第七版),同时选用了该书配套的由邓冠铁等人翻译的中文版教材,在2005级、2006级、2007级进行了双语教学。经过3年的教学实践,笔者认为,这本教材内容涵盖较多、习题量较大,但选材和习题配置与现行中文教材差异较大。鉴于国内目前尚未见到同类英文教材,我们借鉴国外相关教材等英文原版编写了《复

变函数论》双语教材并于 2009 年出版。经过试用，效果良好，故再次出版。第二版修正了原来的一些不当之处，并增加了习题解答提示。

本教材共 6 章，主要内容包括复数与复变函数、解析函数、复变函数的积分、级数、留数及其应用和共形映射等，较全面、系统地介绍了复变函数的基础知识。内容处理上重点突出、叙述简明，每节末附有适量习题供读者选用，适合高等师范院校数学系及普通综合性大学数学系高年级学生使用。

限于编者水平有限，书中不足之处，敬请读者批评指正。

编 者

2014 年 10 月

Contents

[复变函数论]

前言

Chapter I Complex Numbers and Functions

1 Complex Numbers	1
1.1 Complex Number Field	1
1.2 Complex Plane	2
1.3 Modulus, Conjugation, Argument, Polar Representation ...	3
1.4 Powers and Roots of Complex Numbers	6
Exercises	9
2 Regions in the Complex Plane	13
2.1 Some Basic Concept	13
2.2 Domain and Jordan Curve	15
Exercises	17
3 Functions of a Complex Variable	18
3.1 The Concept of Functions of a Complex Variable	18
3.2 Limits and Continuous	19
Exercises	21
4 The Extended Complex Plane and the Point at Infinity	22
4.1 The Spherical Representation, the Extended Complex Plane	22
4.2 Some Concepts in the Extended Complex Plane	23
Exercises	24

Chapter II Analytic Functions

1 The Concept of the Analytic Function	25
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1.1	The Derivative of the Functions of a Complex Variable	25
1.2	Analytic Functions	29
	Exercises	30
2	Cauchy-Riemann Equations	30
	Exercises	35
3	Elementary Functions	36
3.1	The Exponential Function	36
3.2	Trigonometric Functions	37
3.3	Hyperbolic Functions	39
	Exercises	40
4	Multi-Valued Functions	42
4.1	The Logarithmic Function	42
4.2	Complex Power Functions	44
4.3	Inverse Trigonometric and Hyperbolic Functions	46
	Exercises	48

Chapter III Complex Integration

1	The Concept of Contour Integrals	50
1.1	Integral of a Complex Function over a Real Interval	50
1.2	Contour Integrals	51
	Exercises	56
2	Cauchy-Goursat Theorem	60
2.1	Cauchy Theorem	60
2.2	Cauchy Integral Formula	64
2.3	Derivatives of Analytic Functions	66
2.4	Liouville's Theorem and the Fundamental Theorem of Algebra	67
	Exercises	68
3	Harmonic Functions	71
	Exercises	73

Chapter IV Series

1	Basic Properties of Series	74
1.1	Convergence of Sequences	74

1.2	Convergence of Series	76
1.3	Uniform convergence	78
	Exercises	79
2	Power Series	80
	Exercises	84
3	Taylor Series	85
	Exercises	89
4	Laurent Series	93
	Exercises	95
5	Zeros of an Analytic Functions and Uniquely Determined Analytic Functions	97
5.1	Zeros of Analytic Functions	97
5.2	Uniquely Determined Analytic Functions	101
5.3	Maximum Modulus Principle	103
	Exercises	104
6	The Three Types of Isolated Singular Points at a Finite Point	106
	Exercises	112
7	The Three Types of Isolated Singular Points at a Infinite Point	113
	Exercises	114

Chapter V Calculus of Residues

1	Residues	115
1.1	Residues	115
1.2	Cauchy's Residue Theorem	117
1.3	The Calculus of Residue	119
	Exercises	126
2	Applications of Residue	128
2.1	The Type of Definite Integral $\int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta$	128
2.2	The Type of Improper Integral $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx$	130
2.3	The Type of Improper Integral $\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)} \sin ax dx$ or $\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)} \cos ax dx$	132

Exercises	133
3 Argument Principle	136
Exercises	138
Chapter VI Conformal Mappings	
1 Analytic Transformation	140
1.1 Preservation of Domains of Analytic Transformation	140
1.2 Conformality of Analytic Transformation	141
Exercises	145
2 Rational Functions	145
2.1 Polynomials	145
2.2 Rational Functions	146
Exercises	148
3 Fractional Linear Transformations	149
Exercises	154
4 Elementary Conformal Mappings	155
Exercises	163
5 The Riemann Mapping Theorem	165
Exercises	167
Appendix	
Appendix 1	169
Appendix 2	186
Answers	212
Bibliography	224

Chapter I

Complex Numbers and Functions

The complex function is a function of complex variables. The complex functions is a branch of analytics, it is also called Complex Analysis.

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in the real numbers. For instance, $x^2 = 2$ or $x^2 = 3$ are such equations. However, we also know some equations having no solution in the real numbers, for instance, $x^2 = -1$ or $x^2 = -2$. In this chapter, we define a new kind of numbers where such equations have solutions. We will survey the algebraic and geometric structure of the complex number system.

1 Complex Numbers

1.1 Complex Number Field

Definition 1.1.1 We call the numbers form $z = x + iy$ as complex numbers, in which x and y are all real numbers, i is a number that satisfy $i^2 = -1$ and i is called imaginary unit. We call x and y the real and imaginary parts of z and denote this by

$$\operatorname{Re} z = x, \operatorname{Im} z = y \quad (1.1.1)$$

We notice that $z = x$ is a real numbers if $y = 0$, and $z = iy$ is called pure imaginary number if $x = 0$.

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if they have the same real part and the same imaginary part.

The ordinary laws of arithmetic operations are defined as:

$$\begin{aligned} (x_1 + iy_1) \pm (x_2 + iy_2) &= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ \frac{(x_1 + iy_1)}{(x_2 + iy_2)} &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \end{aligned} \quad (1.1.2)$$

As a special case the reciprocal of a complex number $z=x+iy \neq 0$ is given by

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}$$

From the discussion above, we conclude that the set \mathbf{C} of all complex numbers becomes a field, called the field of complex numbers, or the complex field. We may consider \mathbf{R} as a subset of \mathbf{C} .

1.2 Complex Plane

For mapping: $\mathbf{C} \rightarrow \mathbf{R}^2$: $z=x+iy \mapsto (x, y)$ then built a one-to-one correspondence between the set of complex numbers and the plane \mathbf{R}^2 .

With respect to a given rectangular coordinate system in a plane, the complex number $z=x+iy$ can be represented by the point with coordinates (x, y) . The first coordinate axis (x -axis) takes the name of real axis, and the second coordinate axis (y -axis) is called the imaginary axis. The plane itself is referred to as the complex plane.

It is natural to associate any nonzero complex number $z=x+iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z . The number, the point, and the vector will be denoted by the same letter z .

According to the definition of the sum of two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, the number z_1+z_2 corresponds to the point (x_1+x_2, y_1+y_2) . It also corresponds to a vector with those coordinates as its components. Hence z_1+z_2 may be obtained as shown in Fig. 1. The difference $z_1-z_2=z_1+(-z_2)$ corresponds to the sum of the vectors for z_1 and $-z_2$ (Fig. 2).

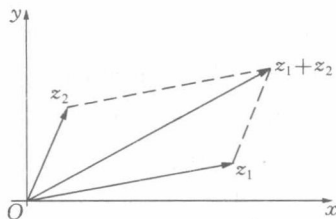


Fig. 1

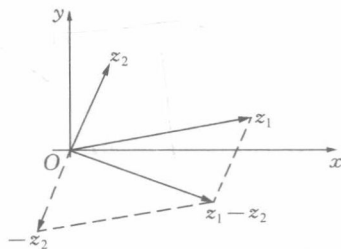


Fig. 2

1.3 Modulus, Conjugation, Argument, Polar Representation

Definition 1.1.2 If $z=x+iy$ then we define

$$|z| = \sqrt{x^2 + y^2} \quad (1.1.3)$$

to be the absolute value of z .

If we think of z as a point in the plane (x, y), then $|z|$ is the length of the line segment from the origin to z . It reduces to the usual absolute value in the real number system when $y=0$.

Theorem 1.1.1 The absolute value of a complex number satisfies the following properties. If z_1, z_2, z are complex numbers, then

$$|-z| = |z| \quad (1.1.4)$$

$$|z_1 z_2| = |z_1| |z_2|, \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad (1.1.5)$$

$$|z_1 \pm z_2| \leq |z_1| + |z_2|, \quad ||z_1| - |z_2|| \leq |z_1 \pm z_2| \quad (1.1.6)$$

(1.1.6) is called the triangle inequality because, if we represent z_1 and z_2 in the plane, (1.1.6) says that the length of one side of the triangle is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line.

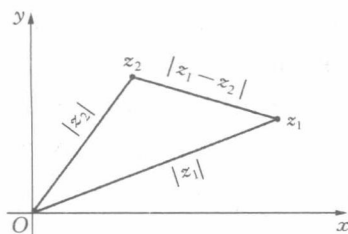


Fig. 3

By mathematical induction we also get:

Theorem 1.1.2 If z_1, z_2, \dots, z_n are complex numbers then we have

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n| \quad (1.1.7)$$

Definition 1.1.3 The complex conjugate, or simply the conjugate, of a complex number $z=x+iy$ is defined as the complex number $x-iy$ and is denoted by \bar{z} ; that is $\bar{z}=x-iy$.

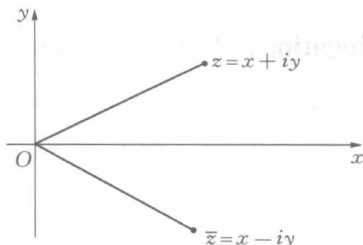


Fig. 4

The point z and its conjugate \bar{z} lie symmetrically with respect to the real axis. This is also easy; in fact, is the point obtained by reflecting z across the x -axis (i. e., the real axis). A number is real if and only if it is equal to its conjugate.

Theorem 1.1.3 The complex conjugate of a complex number satisfies the following properties. If z_1, z_2, z are complex numbers, then

$$\operatorname{Re}z = \frac{z + \bar{z}}{2}, \operatorname{Im}z = \frac{z - \bar{z}}{2i} \quad (1.1.8)$$

$$z\bar{z} = |z|^2, |z| = |\bar{z}|, \overline{\bar{z}} = z \quad (1.1.9)$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2 \quad (1.1.10)$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} (z_2 \neq 0) \quad (1.1.11)$$

Let $(x, y) = x + iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates (r, θ) :

$$x = r \cos \theta \quad y = r \sin \theta \quad (1.1.12)$$

Hence we can write $z = (x, y) = x + iy = r(\cos\theta + i\sin\theta)$. In this trigonometric form of a complex number r is always ≥ 0 and equal to the modulus $|z|$.

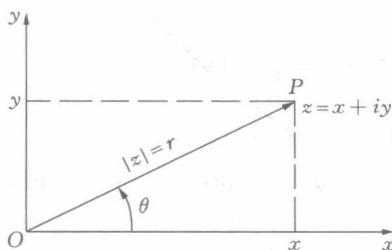


Fig. 5

Definition 1.1.4 The polar angle θ is called the argument of the complex

number, and we denote it by $\text{Arg}z$. The principal value of $\text{Arg}z$, denoted by $\text{arg}z$, is that unique value θ such that $-\pi < \theta \leq \pi$. Note that

$$\text{Arg}z = \{\text{arg}z + 2n\pi; n = 0, \pm 1, \pm 2, \dots\}$$

Simply, we write

$$\text{Arg}z = \text{arg}z + 2n\pi (n = 0, \pm 1, \pm 2, \dots) \tag{1.1.13}$$

Also, when z is a negative real number, $\text{arg}z$ has value π , not $-\pi$.

$$\text{arg}z = \begin{cases} \arctan \frac{y}{x} \dots\dots\dots (x > 0) \\ \frac{\pi}{2} \dots\dots\dots (x = 0, y > 0) \\ \arctan \frac{y}{x} + \pi \dots\dots\dots (x < 0, y \geq 0) \\ \arctan \frac{y}{x} - \pi \dots\dots\dots (x < 0, y < 0) \\ -\frac{\pi}{2} \dots\dots\dots (x = 0, y < 0) \end{cases} \tag{1.1.14}$$

where $-\frac{\pi}{2} < \arctan \frac{y}{x} < \frac{\pi}{2}$.

We list some important identity involving arguments:

$$\text{Arg}(z_1 z_2) = \text{Arg}z_1 + \text{Arg}z_2 \tag{1.1.15}$$

$$\text{Arg}(z_2^{-1}) = -\text{Arg}z_2 \tag{1.1.16}$$

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}z_1 - \text{Arg}z_2 \tag{1.1.17}$$

Example 1 Compute $\text{Arg}(2-2i)$ and $\text{Arg}(-3+4i)$.

$$\text{Arg}(2-2i) = \text{arg}(2-2i) + 2n\pi = \arctan \frac{-2}{2} + 2n\pi = -\frac{\pi}{4} + 2n\pi$$

$$\text{Arg}(-3+4i) = \text{arg}(-3+4i) + 2n\pi = \arctan \frac{4}{-3} + \pi + 2n\pi = (2n+1)\pi - \arctan \frac{4}{3}$$

Example 2 To find the principal argument $\text{arg}z$ when

$$z = \frac{-2}{1 + \sqrt{3}i}$$

Observe that

$$\text{Arg}z = \text{Arg}(-2) - \text{Arg}(1 + \sqrt{3}i)$$

Since

$$\arg(-2) = \pi, \arg(1 + \sqrt{3}i) = \frac{\pi}{3}$$

One value of $\text{Arg}z$ is $\frac{2\pi}{3}$; and, because $\frac{2\pi}{3}$ is between $-\pi$ and π , we find that $\arg z = \frac{2\pi}{3}$.

1.4 Powers and Roots of Complex Numbers

(1) Powers

Definition 1.1.5 We define the expression $e^{i\theta}$ to be

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (\text{Euler's formula}) \quad (1.1.18)$$

where θ is to be measured in radians. Thus $e^{i\theta}$ is a complex number.

It enables us to write the polar form of a complex number in exponential form as

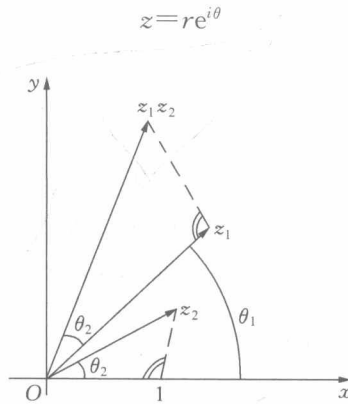


Fig. 6

Theorem 1.1.4

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} \quad (1.1.19)$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad (1.1.20)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \quad (1.1.21)$$

$$z^{-1} = \frac{1}{r} e^{-i\theta} \quad (1.1.22)$$

Expressions (1.1.19), (1.1.20), (1.1.21), and (1.1.22) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x .

Definition 1.1.6 We define the powers of $z=re^{i\theta}$ is

$$z^n = r^n e^{in\theta} \quad (n=0, \pm 1, \pm 2, \dots) \quad (1.1.23)$$

For $r=1$ we obtain de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n=0, \pm 1, \pm 2, \dots) \quad (1.1.24)$$

Which provides an extremely simple way to express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Example 3 The number $-1-i$ has exponential form

$$-1-i = \sqrt{2} e^{i(-\frac{3}{4}\pi)} \quad (1.1.25)$$

Expression (1.1.25) is only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$-1-i = \sqrt{2} e^{i(-\frac{3}{4}\pi + 2n\pi)} \quad (n=0, \pm 1, \pm 2, \dots)$$

Example 4 Put $(\sqrt{3}+i)^7$ in rectangular form.

We write

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3}+i)$$

(2) Square Roots

To find the n th root of a complex number z we have to solve the equation

$$w^n = z \quad (1.1.26)$$

Suppose that $z \neq 0$, $z=re^{i\theta}$, $w=\rho e^{i\phi}$. Then (1.1.26) takes the form

$$\rho^n e^{in\phi} = re^{i\theta}$$

This equation is certainly fulfilled if $\rho^n=r$ and $n\phi=\theta+2k\pi$. Hence we obtain the root

$$w = \sqrt[n]{r} e^{\frac{\theta+2k\pi}{n}}, \quad k=0, \pm 1, \pm 2, \dots$$

However, only the values $k=0, 1, 2, \dots, n-1$ give different value of z .

Definition 1.1.7 We define the n th root of a complex number z is

$$w = \sqrt[n]{r} e^{\frac{\theta+2k\pi}{n}}, \quad k=0, 1, 2, \dots, n-1 \quad (1.1.27)$$

There are n th roots of any complex number $z \neq 0$. They have the same modulus, and their arguments are equally spaced.

Geometrically, the n th roots are the vertices of a regular polygon with n sides.

Example 5 Determine the n th roots of unity.

We write

$$1 = 1e^{i0}$$

and find that

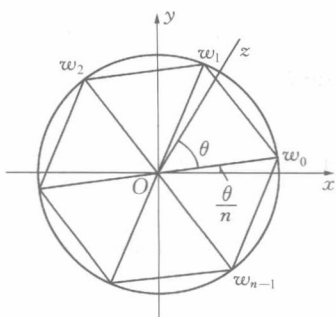


Fig. 7

$$1^{1/n} = \sqrt[n]{1} e^{i \frac{0+2k\pi}{n}} = e^{i \frac{2k\pi}{n}} \quad (k=0, 1, 2, \dots, n-1)$$

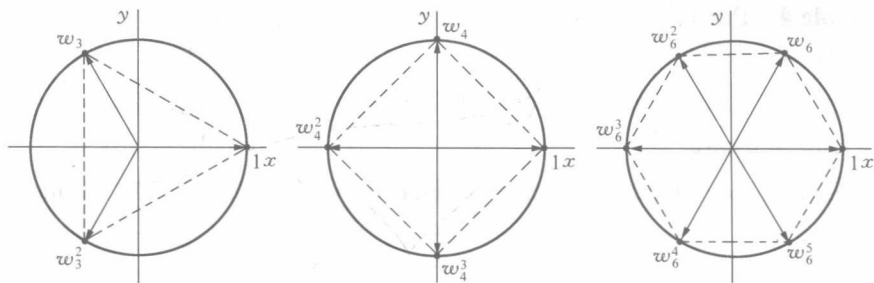


Fig. 8

Example 6 Find the value of $\sqrt[4]{1+i}$.

Because $1+i = \sqrt{2} e^{i\frac{\pi}{4}}$. So that $\sqrt[4]{1+i} = \sqrt[8]{2} e^{i\frac{\pi+2k\pi}{4}} \quad (k=0, 1, 2, 3)$.

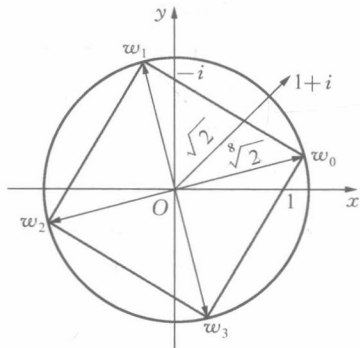


Fig. 9