山东省精品课程教材　山东省双语示范课程教材

FUNCTIONS OF COMPLEX VARIABLES

复变函数论

第二版

Ma Lixin

马立新（编）

CHINA AGRICULTURE PRESS

中国农业出版社

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图书在版编目（CIP）数据

复变函数论／马立新编．—2版．—北京：中国农业出版社，2014.12

ISBN 978-7-109-19726-8

Ⅰ.①复…　Ⅱ.①马…　Ⅲ.①复变函数-高等学校-教材　Ⅳ.①O174.5

中国版本图书馆CIP数据核字（2014）第251917号

中国农业出版社出版

（北京市朝阳区麦子店街18号楼）

（邮政编码 100125）

责任编辑　刘明昌

北京中科印刷有限公司印刷　　新华书店北京发行所发行

2014年12月第2版　　2014年12月第2版北京第1次印刷

开本：720mm×960mm　1/16　　印张：14.5

字数：260千字

定价：35.00元

前　言

伴随着我国高等教育改革形势的发展，高等教育的人才培养模式和教学方式以及教学方法正在发生重大变化。教育部于2001年在《关于加强高等学校本科教学工作提高教学质量的若干意见》（教高【2001】4号）中，明确要求高校要积极开展公共课和专业课教学双语教学的研究和实践。自2005年开始，德州学院采取一系列措施，开展这项创新教育改革活动，推出第一批英汉双语专业课程教学，《复变函数论》即是其中之一。对学生调查的结果及学校组织的听课评议结果均反映良好，其特色得到了山东省课程建设委员会专家的充分肯定，2009年被评为山东省首批双语示范课程、2012年被评为省级精品课程。

对于双语教学，选一本合适的教材极其重要。由于教育体制不同，我国高校复变函数论教材与英美原版教材差别较大，很难找到完全适合我国开展《复变函数论》双语教学的英文原版教材。自2006年起，德州学院选用了James Ward Brown & Ruel V. Churchill编写的英文原版教材Complex Variables and Applications（第七版），同时选用了该书配套的由邓冠铁等人翻译的中文版教材，在2005级、2006级、2007级进行了双语教学。经过3年的教学实践，笔者认为，这本教材内容涵盖较多、习题量较大，但选材和习题配置与现行中文教材差异较大。鉴于国内目前尚未见到同类英文教材，我们借鉴国外相关教材等英文原版编写了《复

变函数论》双语教材并于 2009 年出版。经过试用，效果良好，故再次出版。第二版修正了原来的一些不当之处，并增加了习题解答提示。

本教材共 6 章，主要内容包括复数与复变函数、解析函数、复变函数的积分、级数、留数及其应用和共形映射等，较全面、系统地介绍了复变函数的基础知识。内容处理上重点突出、叙述简明，每节末附有适量习题供读者选用，适合高等师范院校数学系及普通综合性大学数学系高年级学生使用。

限于编者水平有限，书中不足之处，敬请读者批评指正。

编　者

2014 年 10 月

Contents

[复变函数论]

Chapter Ⅰ

Complex Numbers and Functions

The complex function is a function of complex variables. The complex functions is a branch of analytics, it is also called Complex Analysis.

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in the real numbers. For instance, $x^2=2$ or $x^2=3$ are such equations. However, we also know some equations having no solution in the real numbers, for instance, $x^2=-1$ or $x^2=-2$. In this chapter, we define a new kind of numbers where such equations have solutions. We will survey the algebraic and geometric structure of the complex number system.

1 Complex Numbers

1.1 Complex Number Field

Definition 1.1.1 We call the numbers form $z=x+iy$ as complex numbers, in which x and y are all real numbers, i is a number that satisfy $i^2=-1$ and i is called imaginary unit. We call x and y the real and imaginary parts of z and denote this by

$$\operatorname{Re} z=x,\ \operatorname{Im} z=y \tag{1.1.1}$$

We notice that $z=x$ is a real numbers if $y=0$, and $z=iy$ is called pure imaginary number if $x=0$.

Two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are equal if and only if they have the same real part and the same imaginary part.

The ordinary laws of arithmetic operations are defined as:

$$(x_1+iy_1)\pm(x_2+iy_2)=(x_1\pm x_2)+i(y_1\pm y_2)$$

$$(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$

$$\frac{(x_1+iy_1)}{(x_2+iy_2)}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2} \tag{1.1.2}$$

As a special case the reciprocal of a complex number $z=x+iy\neq 0$ is given by

$$\frac{1}{x+iy}=\frac{x}{x^2+y^2}+i\,\frac{-y}{x^2+y^2}$$

From the discussion above, we conclude that the set **C** of all complex numbers becomes a field, called the field of complex numbers, or the complex field. We may consider **R** as a subset of **C**.

1.2 Complex Plane

For mapping: $\mathbf{C}\to\mathbf{R}^2$: $z=x+iy\mapsto(x, y)$ then built a one-to-one correspondence between the set of complex numbers and the plane $\mathbf{R}^2$.

With respect to a given rectangular coordinate system in a plane, the complex number $z=x+iy$ can be represented by the point with coordinates (x, y). The first coordinate axis (x-axis) takes the name of real axis, and the second coordinate axis (y-axis) is called the imaginary axis. The plane itself is referred to as the complex plane.

It is natural to associate any nonzero complex number $z=x+iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector$z$. The number, the point, and the vector will be denoted by the same letter z.

According to the definition of the sum of two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, the number z_1+z_2 corresponds to the point(x_1+x_2, y_1+y_2). It also corresponds to a vector with those coordinates as its components. Hence z_1+z_2 may be obtained as shown in Fig. 1. The difference $z_1-z_2=z_1+(-z_2)$ corresponds to the sum of the vectors for $z_1$and $-z_2$(Fig. 2).

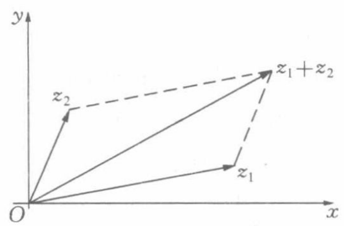

Fig. 1

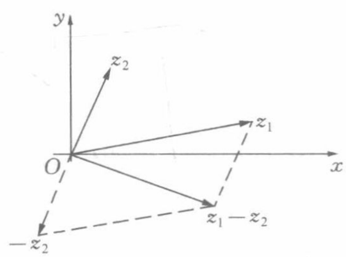

Fig. 2

1.3 Modulus, Conjugation, Argument, Polar Representation

Definition 1.1.2 If $z=x+iy$ then we define

$$|z|=\sqrt{x^2+y^2} \tag{1.1.3}$$

to be the absolute value of z.

If we think of z as a point in the plane(x, y), then $|z|$ is the length of the line segment from the origin to z. It reduces to the usual absolute value in the real number system when$y=0$.

Theorem 1.1.1 The absolute value of a complex number satisfies the following properties. If z_1, z_2, z are complex numbers, then

$$|-z|=|z| \tag{1.1.4}$$

$$|z_1z_2|=|z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|} \tag{1.1.5}$$

$$|z_1\pm z_2|\leqslant|z_1|+|z_2|, \quad ||z_1|-|z_2||\leqslant|z_1\pm z_2| \tag{1.1.6}$$

(1.1.6) is called the triangle inequality because, if we represent z_1 and z_2 in the plane, (1.1.6) says that the length of one side of the triangle is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line.

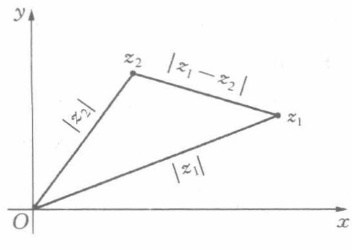

Fig. 3

By mathematical induction we also get:

Theorem 1.1.2 If z_1, z_2, $\cdots$, z_n are complex numbers then we have

$$|z_1+z_2+\cdots+z_n|\leqslant|z_1|+|z_2|+\cdots+|z_n| \tag{1.1.7}$$

Definition 1.1.3 The complex conjugate, or simply the conjugate, of a complex number $z=x+iy$ is defined as the complex number $x-iy$ and is denoted by $\bar{z}$; that is $\bar{z}=x-iy$.

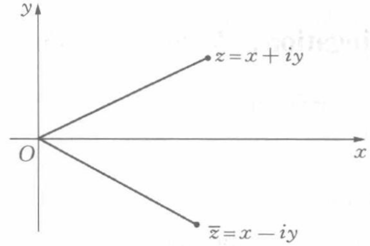

Fig. 4

The point z and its conjugate $\bar{z}$ lie symmetrically with respect to the real axis. This is also easy; in fact, is the point obtained by reflecting z across the x-axis (i. e. , the real axis). A number is real if and only if it is equal to its conjugate.

Theorem 1. 1. 3 The complex conjugate of a complex number satisfies the following properties. If z_1, z_2, z are complex numbers, then

$$\mathrm{Re}z=\frac{z+\bar{z}}{2},\quad \mathrm{Im}z=\frac{z-\bar{z}}{2i} \tag{1.1.8}$$

$$z\bar{z}=|z|^2,\quad |z|=|\bar{z}|,\quad \bar{\bar{z}}=z \tag{1.1.9}$$

$$\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2,\quad \overline{z_1-z_2}=\bar{z}_1-\bar{z}_2 \tag{1.1.10}$$

$$\overline{z_1 z_2}=\bar{z}_1\,\bar{z}_2,\quad \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}\,(z_2\neq 0) \tag{1.1.11}$$

Let $(x, y)=x+iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates(r, θ):

$$x=r\cos\theta \quad y=r\sin\theta \tag{1.1.12}$$

Hence we can write $z=(x, y)=x+iy=r(\cos\theta+i\sin\theta)$. In this trigonometric form of a complex number r is always $\geqslant 0$ and equal to the modulus $|z|$.

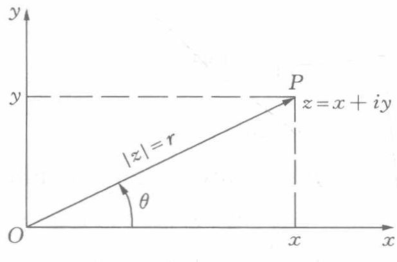

Fig. 5

Definition 1. 1. 4 The polar angle θ is called the argument of the complex number, and we denote it by Arg$z$. The principal value of Arg$z$, denoted by arg$z$, is that unique value θ such that $-\pi<\theta\leqslant\pi$. Note that

$$\operatorname{Arg}z=\{\arg z+2n\pi : n=0,\pm1,\pm2,\cdots\}$$

Simply, we write

$$\operatorname{Arg}z=\arg z+2n\pi(n=0,\ \pm1,\ \pm2,\ \cdots) \tag{1.1.13}$$

Also, when z is a negative real number, arg$z$ has value π, not $-\pi$.

$$\underset{(z\neq0)}{\arg z}=\begin{cases}\arctan\dfrac{y}{x}\cdots\cdots\cdots\cdots\cdots (x>0)\\ \dfrac{\pi}{2}\cdots\cdots\cdots\cdots\cdots\cdots\cdots (x=0,\ y>0)\\ \arctan\dfrac{y}{x}+\pi\cdots\cdots\cdots (x<0,\ y\geqslant0)\\ \arctan\dfrac{y}{x}-\pi\cdots\cdots\cdots (x<0,\ y<0)\\ -\dfrac{\pi}{2}\cdots\cdots\cdots\cdots\cdots\cdots\cdots (x=0,\ y<0)\end{cases} \tag{1.1.14}$$

where $-\dfrac{\pi}{2}<\arctan\dfrac{y}{x}<\dfrac{\pi}{2}$.

We list some important identity involving arguments:

$$\operatorname{Arg}(z_1z_2)=\operatorname{Arg}z_1+\operatorname{Arg}z_2 \tag{1.1.15}$$

$$\operatorname{Arg}(z_2^{-1})=-\operatorname{Arg}z_2 \tag{1.1.16}$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right)=\operatorname{Arg}z_1-\operatorname{Arg}z_2 \tag{1.1.17}$$

Example 1 Compute Arg$(2-2i)$ and Arg$(-3+4i)$.

$$\operatorname{Arg}(2-2i)=\arg(2-2i)+2n\pi=\arctan\frac{-2}{2}+2n\pi=-\frac{\pi}{4}+2n\pi$$

$$\operatorname{Arg}(-3+4i)=\arg(-3+4i)+2n\pi=\operatorname{argtan}\frac{4}{-3}+\pi+2n\pi=(2n+1)\pi-\arctan\frac{4}{3}$$

Example 2 To find the principal argument arg$z$ when

$$z=\frac{-2}{1+\sqrt{3i}}$$

Observe that

$$\operatorname{Arg}z=\operatorname{Arg}(-2)-\operatorname{Arg}(1+\sqrt{3i})$$

Since

$$\arg(-2)=\pi, \arg(1+\sqrt{3i})=\frac{\pi}{3}$$

One value of Arg$z$ is $\frac{2\pi}{3}$; and, because $\frac{2\pi}{3}$ is between $-\pi$ and π, we find that arg$z=\frac{2\pi}{3}$.

1.4 Powers and Roots of Complex Numbers

(1) Powers

Definition 1.1.5 We define the expression $e^{i\theta}$ to be

$$e^{i\theta}=\cos\theta+i\sin\theta \text{ (Euler's formula)} \quad (1.1.18)$$

where θ is to be measured in radians. Thus $e^{i\theta}$ is a complex number.

It enables us to write the polar form of a complex number in exponential form as

$$z=re^{i\theta}$$

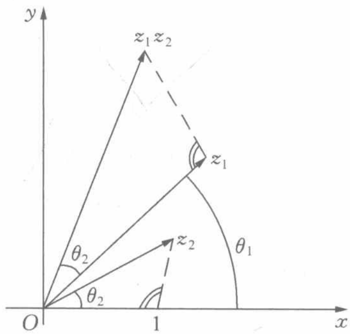

Fig. 6

Theorem 1.1.4

$$e^{i\theta_1}e^{i\theta_2}=e^{i(\theta_1+\theta_2)} \quad (1.1.19)$$

$$z_1z_2=r_1r_2e^{i(\theta_1+\theta_2)} \quad (1.1.20)$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \quad (1.1.21)$$

$$z^{-1}=\frac{1}{r}e^{-i\theta} \quad (1.1.22)$$

Expressions (1.1.19), (1.1.20), (1.1.21), and (1.1.22) are, of course, easily remembered by applying the usual algebraic rules for real numbers and e^x.

Definition 1.1.6 We define the powers of $z=re^{i\theta}$ is

$$z^n=r^n e^{in\theta}(n=0, \pm1, \pm2, \cdots) \tag{1.1.23}$$

For $r=1$ we obtain de Moivre's formula

$$(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta (n=0, \pm1, \pm2, \cdots) \tag{1.1.24}$$

Which provides an extremely simple way to express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Example 3 The number $-1-i$ has exponential form

$$-1-i=\sqrt{2}e^{i(-\frac{3}{4}\pi)} \tag{1.1.25}$$

Expression (1.1.25) is only one of an infinite number of possibilities for the exponential from of $-1-i$:

$$-1-i=\sqrt{2}e^{i(-\frac{3}{4}\pi+2n\pi)} (n=0, \pm1, \pm2, \cdots)$$

Example 4 Put $(\sqrt{3}+i)^7$ in rectangular form.

We write

$$(\sqrt{3}+i)^7=(2e^{i\pi/6})^7=2^7e^{i7\pi/6}=(2^6e^{i\pi})(2e^{i\pi/6})=-64(\sqrt{3}+i)$$

(2) Square Roots

To find the n th root of a complex number z we have to solve the equation

$$w^n=z \tag{1.1.26}$$

Suppose that $z\neq0$, $z=re^{i\theta}$, $w=\rho e^{i\phi}$. Then (1.1.26) takes the form

$$\rho^n e^{in\phi}=re^{i\theta}$$

This equation is certainly fulfilled if $\rho^n=r$ and $n\phi=\theta+2k\pi$. Hence we obtain the root

$$w=\sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k=0, \pm1, \pm2, \cdots$$

However, only the values $k=0, 1, 2, \cdots, n-1$ give different value of z.

Definition 1.1.7 We define the n th root of a complex number z is

$$w=\sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k=0, 1, 2, \cdots, n-1 \tag{1.1.27}$$

There are n th roots of any complex number $z\neq0$. They have the same modulus, and their arguments are equally spaced.

Geometrically, the n th roots are the vertices of a regular polygon with n sides.

Example 5 Determine the n th roots of unity.

We write

$$1=1e^{i0}$$

and find that

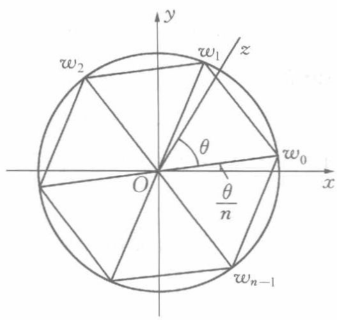

Fig. 7

$$1^{\frac{1}{n}}=\sqrt[n]{1}\,e^{i\frac{0+2k\pi}{n}}=e^{i\frac{2k\pi}{n}}\ (k=0,\ 1,\ 2,\ \cdots,\ n-1)$$

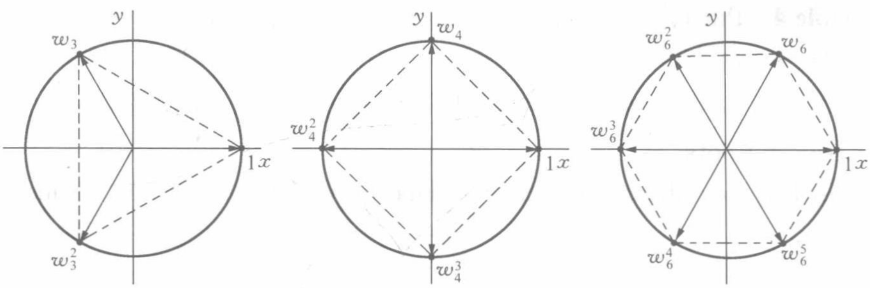

Fig. 8

Example 6 Find the value of $\sqrt[4]{1+i}$.

Because $1+i=\sqrt{2}\,e^{i\frac{\pi}{4}}$. So that $\sqrt[4]{1+i}=\sqrt[8]{2}\,e^{i\frac{\frac{\pi}{4}+2k\pi}{4}}\ (k=0,\ 1,\ 2,\ 3)$.

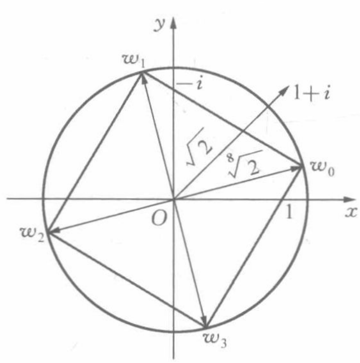

Fig. 9