Fourier Integrals in Classical Analysis

经典分析中的傅立叶积分

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Fourier Integrals in Classical Analysis is an advanced monograph concerned with modern treatments of central problems in harmonic analysis. The main theme of the book is the interplay between ideas used to study the propagation of singularities for the wave equation and their counterparts in classical analysis. Using microlocal analysis, the author, in particular, studies problems involving maximal functions and Riesz means using the so-called half-wave operator.

This self-contained book starts with a rapid review of important topics in Fourier analysis. The author then presents the necessary tools from microlocal analysis, and goes on to give a proof of the sharp Weyl formula which he then modifies to give sharp estimates for the size of eigenfunctions on compact manifolds. Finally, at the end, the tools that have been developed are used to study the regularity properties of Fourier integral operators, culminating in the proof of local smoothing estimates and their applications to singular maximal theorems in two and more dimensions.

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To my family

Preface

Except for minor modifications, this monograph represents the lecture notes of a course I gave at UCLA during the winter and spring quarters of 1991. My purpose in the course was to present the necessary background material and to show how ideas from the theory of Fourier integral operators can be useful for studying basic topics in classical analysis, such as oscillatory integrals and maximal functions. The link between the theory of Fourier integral operators and classical analysis is of course not new, since one of the early goals of microlocal analysis was to provide variable coefficient versions of the Fourier transform. However, the primary goal of this subject was to develop tools for the study of partial differential equations and, to some extent, only recently have many classical analysts realized its utility in their subject. In these notes I attempted to stress the unity between these two subjects and only presented the material from microlocal analysis which would be needed for the later applications in Fourier analysis. I did not intend for this course to serve as an introduction to microlocal analysis. For this the reader should be referred to the excellent treatises of Hörmander [5], [7] and Treves [1].

In addition to these sources, I also borrowed heavily from Stein [4]. His work represents lecture notes based on a course which he gave at Princeton while I was his graduate student. As the reader can certainly tell, this course influenced me quite a bit and I am happy to acknowledge my indebtedness. My presentation of the overlapping material is very similar to his, except that I chose to present the material in the chapter on oscillatory integrals more geometrically, using the cotangent bundle. This turns out to be useful in dealing with Fourier analysis on manifolds and it also helps to motivate some results concerning Fourier integral operators, in particular the local smoothing estimates at the end of the monograph.

Roughly speaking, the material is organized as follows. The first two chapters present background material on Fourier analysis and stationary phase that will be used throughout. The next chapter deals with non-homogeneous oscillatory integrals. It contains the L^2 restriction theorem for the Fourier transform, estimates for Riesz means in $\mathbb{R}^n$, and Bourgain's circular maximal theorem. The goal of the rest of the monograph is mainly to develop generalizations of these results. The first step in this direction is to present some basic background material from the theory of pseudo-differential operators, emphasizing the role of stationary phase. After the chapter on pseudo-differential operators comes one dealing with the sharp Weyl formula of Hörmander [4], Avakumovič [1], and Levitan [1]. I followed the exposition in Hörmander's paper, except that the Tauberian condition in the proof of the Weyl formula is stated in terms of L^∞ estimates for eigenfunctions. In the next chapter, this slightly different point of view is used in generalizing some of the earlier results from Fourier analysis in $\mathbb{R}^n$ to the setting of compact manifolds. Finally, the last two chapters are concerned with Fourier integral operators. First, some background material is presented and then the mapping properties of Fourier integral operators are investigated. This is all used to prove some recent local smoothing estimates for Fourier integral operators, which in turn imply variable coefficient versions of Stein's spherical maximal theorem and Bourgain's circular maximal theorem.

It is a pleasure to express my gratitude to the many people who helped me in preparing this monograph. First, I would like to thank everyone who attended the course for their helpful comments and suggestions. I am especially indebted to D. Grieser, A. Iosevich, J. Johnsen, and H. Smith who helped me both mathematically and in proofreading. I am also grateful to M. Cassorla and R. Strichartz for their thorough critical reading of earlier versions of the manuscript. Lastly, I would like to thank all of my collaborators for the important role they have played in the development of many of the central ideas in this course. In this regard, I am particularly indebted to A. Seeger and E. M. Stein.

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Sherman Oaks *C. D. Sogge*

Contents

Chapter 0

Background

The purpose of this chapter and the next is to present the background material that will be needed. The topics are standard and a more thorough treatment can be found in many excellent sources, such as Stein [2] and Stein and Weiss [1] for the first half and Hörmander [7, Vol. 1] for the second.

We start out by rapidly going over basic results from real analysis, including standard theorems concerning the Fourier transform in $\mathbb{R}^n$ and Caldéron-Zygmund theory. We then apply this to prove the Hardy-Littlewood-Sobolev inequality. This theorem on fractional integration will be used throughout and we shall also present a simple argument showing how the n-dimensional theorem follows from the original one-dimensional inequality of Hardy and Littlewood. This type of argument will be used again and again. Finally, in the last two sections we give the definition of the wave front set of a distribution and compute the wave front sets of distributions which are given by oscillatory integrals. This will be our first encounter with the cotangent bundle and, as the monograph progresses, this will play an increasingly important role.

0.1. Fourier Transform

Given $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx. \tag{0.1.1}$$

Given $h \in \mathbb{R}^n$, let $(\tau_h f)(x) = f(x+h)$. Notice that $\tau_{-h} e^{-i\langle \cdot, \xi \rangle} = e^{i\langle h,\xi\rangle} e^{-i\langle \cdot,\xi\rangle}$ and so

$$(\tau_h f)^\wedge(\xi) = e^{i\langle h,\xi\rangle} \hat{f}(\xi). \tag{0.1.2}$$

In a moment, we shall see that we can invert (0.1.1) (for appropriate f) and that we have the formula

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi. \tag{0.1.3}$$

Thus, the Fourier transform decomposes a function into a continuous sum of characters (eigenfunctions for translations).

Before turning to Fourier's inversion formula (0.1.3), let us record some elementary facts concerning the Fourier transform of L^1 functions.

Theorem 0.1.1:

(1) $\|\hat{f}\|_\infty \le \|f\|_1$.
(2) *If $f \in L^1$, then $\hat{f}$ is uniformly continuous.*

Theorem 0.1.2 (Riemann-Lebesgue): *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi) \to 0$ as $\xi \to \infty$, and, hence, $\hat{f} \in C_0(\mathbb{R}^n)$.*

Theorem 0.1.1 follows directly from the definition (0.1.1). To prove Theorem 0.1.2, one first notices from an explicit calculation that the result holds when f is the characteristic function of a cube. From this one derives Theorem 0.1.2 via a limiting argument.

Even though $\hat{f}$ is in C_0, the integral (0.1.3) will not converge for general $f \in L^1$. However, for a dense subspace we shall see that the integral converges absolutely and that (0.1.3) holds.

Definition 0.1.3: The set of Schwartz-class functions, $\mathcal{S}(\mathbb{R}^n)$, consists of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\sup_x |x^\gamma \partial^\alpha \phi(x)| < \infty, \tag{0.1.4}$$

for all multi-indices α, γ.[1]

We give $\mathcal{S}$ the topology arising from the semi-norms (0.1.4). This makes $\mathcal{S}$ a Fréchet space. Notice that the set of all compactly supported C^∞ functions, $C_0^\infty(\mathbb{R}^n)$, is contained in $\mathcal{S}$.

Let $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Then we have:

Theorem 0.1.4: *If $\phi \in \mathcal{S}$, then the Fourier transform of $D_j\phi$ is $\xi_j \hat{\phi}(\xi)$. Also, the Fourier transform of $x_j\phi$ is $-D_j\hat{\phi}$.*

[1] Here $\alpha = (\alpha_1, \dots, \alpha_n), \gamma = (\gamma_1, \dots, \gamma_n)$ and $x^\gamma = x^{\gamma_1} \cdots x^{\gamma_n}$, $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

Proof: To prove the second assertion we differentiate (0.1.1) to obtain

$$D_j\hat{\phi}(\xi) = \int e^{-i\langle x,\xi\rangle}(-x_j)\phi(x)\,dx,$$

since the integral converges uniformly. If we integrate by parts, we see that

$$\xi_j\hat{\phi}(\xi) = \int -D_j e^{-i\langle x,\xi\rangle}\cdot\phi(x)\,dx = \int e^{-i\langle x,\xi\rangle}D_j\phi(x)\,dx,$$

which is the first assertion. ∎

Notice that Theorem 0.1.4 implies the formula

$$\xi^\alpha D^\gamma\hat{\phi}(\xi) = \int e^{-i\langle x,\xi\rangle}D^\alpha\big((-x)^\gamma\phi(x)\big)\,dx. \tag{0.1.5}$$

If we set $C = \int(1+|x|)^{-n-1}\,dx$, then this leads to the estimate

$$\sup_\xi|\xi^\gamma D^\alpha\hat{\phi}(\xi)| \le C\sup_x(1+|x|)^{n+1}\big|D^\gamma\big(x^\alpha\phi(x)\big)\big|. \tag{0.1.6}$$

Inequality (0.1.6) of course implies that the Fourier transform maps $\mathcal{S}$ into itself. However, much more is true:

Theorem 0.1.5: *The Fourier transform $\phi \to \hat{\phi}$ is an isomorphism of $\mathcal{S}$ into $\mathcal{S}$ whose inverse is given by Fourier's inversion formula* (0.1.3).

The proof is based on a couple of lemmas. The first is the multiplication formula for the Fourier transform:

Lemma 0.1.6: *If $f, g \in L^1$ then*

$$\int_{\mathbf{R}^n}\hat{f}g\,dx = \int_{\mathbf{R}^n}f\hat{g}\,dx.$$

The next is a formula for the Fourier transform of Gaussians:

Lemma 0.1.7: $\int_{\mathbf{R}^n}e^{-i\langle x,\xi\rangle}\,e^{-\varepsilon|x|^2/2}\,dx = \big(2\pi/\varepsilon\big)^{n/2}\,e^{-|\xi|^2/2\varepsilon}$.

The first lemma is easy to prove. If we apply (0.1.1) and Fubini's theorem, we see that the left side equals

$$\int\Big\{\int f(y)e^{-i\langle x,y\rangle}\,dy\Big\}g(x)\,dx = \int\Big\{\int e^{-i\langle x,y\rangle}g(x)\,dx\Big\}f(y)\,dy$$
$$= \int\hat{g}f\,dy.$$

It is also clear that Lemma 0.1.7 must follow from the special case where $n = 1$. But

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-t^2/2} e^{-it\tau}\, dt &= e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\tau)^2}\, dt \\ &= e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt \\ &= \sqrt{2\pi} e^{-\tau^2/2}.\end{aligned}$$

In the second step we have used Cauchy's theorem. If we make the change of variables $\varepsilon^{1/2} s = t$ in the last integral, we get the desired result.

Proof of Theorem 0.1.5: We must prove that when $\phi \in \mathcal{S}$,

$$\phi(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{\phi}(\xi)\, d\xi.$$

By the dominated convergence theorem, the right side equals

$$\lim_{\varepsilon \to 0} (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{\phi}(\xi) e^{-\varepsilon|\xi|^2/2}\, d\xi.$$

If we recall (0.1.2), then we see that this equals

$$\lim_{\varepsilon \to 0} (2\pi\varepsilon)^{-n/2} \int \phi(x+y)\, e^{-|y|^2/2\varepsilon}\, dy.$$

Finally, since $(2\pi)^{-n/2} \int e^{-|y|^2/2}\, dy = 1$, it is easy to check that the last limit is $\phi(x)$. ∎

If for $f, g \in L^1$ we define convolution by

$$(f * g)(x) = \int f(x-y) g(y)\, dy,$$

then another fundamental result is:

Theorem 0.1.8: *If $\phi, \psi \in \mathcal{S}$ then*

$$(2\pi)^n \int \phi\overline{\psi}\, dx = \int \hat{\phi}\overline{\hat{\psi}} d\xi, \tag{0.1.7}$$

$$(\phi * \psi)^\wedge(\xi) = \hat{\phi}(\xi)\hat{\psi}(\xi), \tag{0.1.8}$$

$$(\phi\psi)^\wedge(\xi) = (2\pi)^{-n} (\hat{\phi} * \hat{\psi})(\xi). \tag{0.1.9}$$

To prove (0.1.7), set $\chi = (2\pi)^{-n} \overline{\hat{\psi}}$. Then the Fourier inversion formula implies that $\hat{\chi} = \overline{\psi}$. Consequently, (0.1.7) follows from Lemma 0.1.6. We leave the other two formulas as exercises.

We shall now discuss the Fourier transform of more general functions. First, we make a definition.

Definition 0.1.9: The dual space of $\mathcal{S}$ is $\mathcal{S}'$. We call $\mathcal{S}'$ the space of tempered distributions.

Definition 0.1.10: If $u \in \mathcal{S}'$, we define its Fourier transform $\hat{u} \in \mathcal{S}'$ by setting, for all $\phi \in \mathcal{S}$,

$$\hat{u}(\phi) = u(\hat{\phi}). \tag{0.1.10}$$

Notice how Lemma 0.1.6 says that when $u \in L^1$, Definition 0.1.10 agrees with our previous definition of $\hat{u}$. Using Fourier's inversion formula for $\mathcal{S}$, one can check that $u \to \hat{u}$ is an isomorphism of $\mathcal{S}'$. If $u \in L^1$ and $\hat{u} \in L^1$, we conclude that the inversion formula (0.1.3) must hold for almost all x.

Theorem 0.1.11: *If $u \in L^2$ then $\hat{u} \in L^2$ and*

$$\|\hat{u}\|_2^2 = (2\pi)^n \|u\|_2^2 \qquad \text{(Plancherel's theorem).} \tag{0.1.11}$$

Furthermore, Parseval's formula holds whenever $\phi, \psi \in L^2$:

$$\int \phi\overline{\psi}\, dx = (2\pi)^{-n} \int \hat{\phi}\overline{\hat{\psi}}\, dx. \tag{0.1.12}$$

Proof: Choose $u_j \in \mathcal{S}$ satisfying $u_j \to u$ in L^2. Then, by (0.1.7),

$$\|\hat{u}_j - \hat{u}_k\|_2^2 = (2\pi)^n \|u_j - u_k\|_2^2 \to 0.$$

Thus, $\hat{u}_j$ converges to a function v in L^2. But the continuity of the Fourier transform in $\mathcal{S}'$ forces $v = \hat{u}$. This gives (0.1.11), since (0.1.11) is valid for each u_j. Since we have just shown that the Fourier transform is continuous on L^2, (0.1.12) follows from the fact that we have already seen that it holds when ϕ and ψ belong to the dense subspace $\mathcal{S}$. ∎

Since, for $1 \le p \le 2$, $f \in L^p$ can be written as $f = f_1 + f_2$ with $f_1 \in L^1$, $f_2 \in L^2$, it follows from Theorem 0.1.1 and Theorem 0.1.11 that $\hat{f} \in L^2_{\text{loc}}$. A much better result is:

Theorem 0.1.12 (Hausdorff-Young): *Let $1 \le p \le 2$ and define p' by $1/p + 1/p' = 1$. Then, if $f \in L^p$ it follows that $\hat{f} \in L^{p'}$ and*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n/p'} \|f\|_p.$$

Since we have already seen that this result holds for $p = 1$ and $p = 2$, this follows from:

Theorem 0.1.13 (M. Riesz interpolation theorem): *Let T be a linear map from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} \cap L^{q_1}$ satisfying*

$$\|Tf\|_{q_j} \le M_j \|f\|_{p_j}, \quad j = 0, 1, \tag{0.1.13}$$

with $1 \le p_j, q_j \le \infty$. Then, if for $0 < t < 1$, $1/p_t = (1-t)/p_0 + t/p_1, 1/q_t = (1-t)/q_0 + t/q_1$,

$$\|Tf\|_{q_t} \le (M_0)^{1-t} (M_1)^t \|f\|_{p_t}, \quad f \in L^{p_0} \cap L^{p_1}. \tag{0.1.14}$$

Proof: If $p_t = \infty$ the result follows from Hölder's inequality since then $p_0 = p_1 = \infty$. So we shall assume that $p_t < \infty$.

By polarization it then suffices to show that

$$\left| \int Tfg\,dx \right| \le M_0^{1-t} M_1^t \|f\|_{p_t} \|g\|_{q_t'} \tag{0.1.15}$$

when f and g vanish outside of a set of finite measure and take on a finite number of values, that is, $f = \sum_{j=1}^m a_j \chi_{E_j}, g = \sum_{k=1}^N b_k \chi_{F_k}$, with $E_j \cap E_{j'} = \emptyset$ and $F_k \cap F_{k'} = \emptyset$ if $j \ne j'$ and $k \ne k'$ and $|E_j|, |F_k| < \infty$ for all j and k. We may also assume $\|f\|_{p_t}, \|g\|_{q_t'} \ne 0$ and so, if we divide both sides by the norms, it suffices to prove (0.1.15) when $\|f\|_{p_t} = \|g\|_{q_t'} = 1$.

Next, if $a_j = e^{i\theta_j}|a_j|$ and $b_k = e^{i\psi_k}|b_k|$, then, assuming $q_t > 1$, we set

$$f_z = \sum_{j=1}^m |a_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j} \chi_{E_j},$$

$$g_z = \sum_{k=1}^N |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i\psi_k} \chi_{F_k},$$

where $\alpha(z) = (1-z)/p_0 + z/p_1$ and $\beta(z) = (1-z)/q_0 + z/q_1$. If $q_t = 1$ then we modify the definition by taking $g_z \equiv g$. It then follows that $F(z) = \int Tf_z g_z\,dx$ is entire and bounded in the strip $0 \le \mathrm{Re}\,(z) \le 1$. Also, $F(t)$ equals the left side of (0.1.15). Consequently, by the three-lines lemma,[2] we would be done if we could prove

$$|F(z)| \le M_0, \quad \mathrm{Re}(z) = 0,$$

$$|F(z)| \le M_1, \quad \mathrm{Re}(z) = 1.$$

To prove the first inequality, notice that for $y \in \mathbb{R}$, $\alpha(iy) = 1/p_0 + iy(1/p_1 - 1/p_0)$. Consequently,

$$|f_{iy}|^{p_0} = \left| e^{i \arg f} \cdot |f|^{iy(1/p_1 - 1/p_0)} \cdot |f|^{p_t/p_0} \right|^{p_0} = |f|^{p_t}.$$

[2] See, for example, Stein and Weiss [1, p. 180].