高等院校双语教学规划教材

（英文版）

Advanced Mathematics (II)

高等数学（下）

东南大学大学数学教研室　编著

东南大学出版社
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·南京·

内 容 提 要

本书是为响应东南大学国际化需要，根据国家教育部非数学专业数学基础课教学指导分委员会制定的工科类本科数学基础课程教学基本要求，并结合东南大学数学系多年教学改革实践经验编写的全英文教材。全书分为上、下两册，内容包括极限、一元函数微分学、一元函数积分学、常微分方程、级数、向量代数与空间解析几何、多元函数微分学、多元函数积分学、向量场的积分、复变函数等十个章节。

本书可作为高等理工科院校非数学类专业本科生学习高等数学课程的英文教材，也可供其他专业选用和社会读者阅读。

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前　言

本书是为响应东南大学国际化需要，根据国家教育部非数学专业数学基础课教学指导分委员会制定的工科类本科数学基础课程教学基本要求，并结合东南大学数学系多年教学改革实践经验编写的全英文教材。全书分为上、下两册，内容包括极限、一元函数微分学、一元函数积分学、常微分方程、级数、向量代数与空间解析几何、多元函数微分学、多元函数积分学、向量场的积分、复变函数等十个章节。

本书对基本概念的叙述清晰准确，对基本理论的论述简明易懂。在内容处理上依据国内工科类本科数学基础课程教学基本要求，按照现行的国内微积分教材体系结构进行编排，比国外同类教材简洁，理论性更强。同时，本书还兼顾美国教材重视应用、便于自学的特点，例题和习题的选配典型多样，增加了应用内容与相关的实际问题，强调对基本运算能力及理论的实际应用能力的培养。

本教材的内容是工科学生必备的大学数学知识，利用英文编写更有利于学生提高与国际同行专家交流的能力。本书可作为高等理工科院校非数学类专业本科生学习高等数学课程的英文教材，也可供其他专业选用和社会读者阅读。

本书下册共六章，其中第五章由马红铝编写，第六、七、八、九章由陈文彦编写，第十章由刘国华编写。全书由陈文彦统稿。

本书在编写过程中得到了东南大学教务处的大力支持，数学系的王栓宏教授对本教材的编写提出了许多有益的建议，在此一并对他们表示感谢。本书中缺点和错误在所难免，欢迎读者批评指正。

编　者
2014 年 11 月

CONTENTS

Chapter 5 Infinite Series

In Chapter 2 we saw how a wide variety of functions could be approximated by polynomials. A polynomial is a finite sum of terms of the form $a_k x^k$. In this chapter we will discuss what we mean by an infinite sum. The infinite sums we describe are called infinite series. We will discuss the theory of infinite series and will show how it can give us a great deal of information about a wide variety of functions.

§5.1 Infinite Series

§5.1.1 The Concept of Infinite Series

Definition 1 Let $\{u_n\}$ be a sequence of real numbers. Then the formal sum

$$u_1+u_2+\cdots+u_n+\cdots$$

is called an **infinite series**(or, simply, **series**), denoted by $\sum_{n=1}^{\infty} u_n$. The numbers $u_1, u_2, \cdots, u_n, \cdots$ are its **terms** and the number u_n is the ***n*th term** of the series. The **partial sums** of the series are given by

$$S_n = \sum_{k=1}^{n} u_k .$$

The term S_n is called the ***n*th partial sum** of the series. If the sequence of partial sums $\{S_n\}$ converges to S, then we say that the infinite series $\sum_{n=1}^{\infty} u_n$ converges to S and we write

$$\sum_{n=1}^{\infty} u_n = S.$$

Otherwise we say that the series $\sum_{n=1}^{\infty} u_n$ diverges.

Of course, an infinite series need not begin with the term u_1; it could equally well begin with u_2, u_5, u_{99}, or any other term. We shall generally begin series with u_0 or u_1, however.

A series of the form

$$\sum_{n=0}^{\infty} aq^n = a + aq + aq^2 + \cdots + aq^n + \cdots$$

where $a \neq 0$, is called a **geometric series.**

Example 1 Show that a geometric series converges, and $\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$ if $|q|<1$, but diverges if $|q| \geqslant 1$.

Solution Since convergence is defined in terms of partial sums, we look at

$$S_n = \sum_{k=0}^{n-1} aq^k .$$

From elementary algebra, we have

$$S_n = a\frac{1-q^n}{1-q} \quad (q \neq 1).$$

If $|q|<1$, then $\lim_{n\to\infty} q^n = 0$, and thus

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-q}.$$

If $|q|>1$ or $q=-1$, the sequence $\{q^n\}$ diverges, and consequently so does $\{S_n\}$. If $q=1$, $S_n = na$, which grows without bound, and so $\{S_n\}$ diverges.

Example 2 Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$$

converges, and find its sum.

Solution Since

$$\begin{aligned} S_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}, \end{aligned}$$

and $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$, we conclude that the sum is 1.

Example 3 Test for convergence or divergence: $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$.

Solution The observation that

$$\ln\left(1+\frac{1}{n}\right)=\ln(n+1)-\ln n$$

permits us to write the partial sum as

$$S_n=\sum_{k=1}^{n}\ln\left(1+\frac{1}{k}\right)=\sum_{k=1}^{n}(\ln(k+1)-\ln k)=\ln(n+1).$$

Hence

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\ln(n+1)=+\infty,$$

and the series diverges.

§ 5.1.2 Conditions for Convergence

Theorem 1 (The *n*th-Term Test for Divergence) If the series $\sum_{n=1}^{\infty}u_n$ converges, then $\lim_{n\to\infty}u_n=0$. Equivalently, if $\lim_{n\to\infty}u_n\neq 0$ or if $\lim_{n\to\infty}u_n$ does not exist, then the series diverges.

Proof Let S_n be the nth partial sum and $\sum_{n=1}^{\infty}u_n=S$. Note that $u_n=S_n-S_{n-1}$. Since $\lim_{n\to\infty}S_n=\lim_{n\to\infty}S_{n-1}=S$, it follows that

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}(S_n-S_{n-1})=\lim_{n\to\infty}S_n-\lim_{n\to\infty}S_{n-1}=S-S=0.$$

Note that students invariably want to turn Theorem 1 around and make it say that $\lim_{n\to\infty}u_n=0$ implies convergence of $\sum_{n=1}^{\infty}u_n$. The **Harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$$

shows that this is false. Clearly, $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n}=0$. However, the series diverges, as we now show.

Example 4 Show that the harmonic series diverges.

Proof We show that $\{S_n\}$ grows without bound. Consider subsequence $\{S_{2^k}\}$ of $\{S_n\}$:

$$S_{2^1}=S_2=1+\frac{1}{2},$$

$$S_{2^2}=S_4=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1+2\cdot\frac{1}{2},$$

$$S_{2^3}=S_8=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=S_{2^2}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$$

$$>1+2\cdot\frac{1}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=1+3\cdot\frac{1}{2}.$$

We assume that $S_{2^k}>1+k\cdot\frac{1}{2}$, then by the Principle of Mathematical Induction

$$S_{2^{k+1}}=S_{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\cdots+\frac{1}{2^{k+1}}$$

$$>1+k\cdot\frac{1}{2}+\frac{1}{2^{k+1}}+\cdots+\frac{1}{2^{k+1}}=1+(k+1)\cdot\frac{1}{2}.$$

It is clear that by taking k sufficiently large, S_{2^k} grows without bound, and so $\{S_n\}$ diverges. Hence, the harmonic series diverges.

By the nth-Term Test for Divergence, we can see that $\sum_{n=0}^{\infty}(-1)^n$ diverges since the sequence $\{(-1)^n\}$ diverges; and $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$ diverges since

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}\neq 0.$$

Since convergence of a series is defined in terms of the convergence of the sequence of partial sums, any information about the convergence of sequences is useful in discussing series. Of particular importance in this connection is the Cauchy Criterion, which takes the following form.

Theorem 2 (Cauchy Criterion) A series $\sum_{n=1}^{\infty}u_n$ converges if and only if for every $\varepsilon>0$, there is $N\in\mathbf{N}^*$, such that for every $n>N$, for every $p\in\mathbf{N}^*$ implies that

$$|S_{n+p}-S_n|=\left|\sum_{k=n+1}^{n+p}u_k\right|=|u_{n+1}+u_{n+2}+\cdots+u_{n+p}|<\varepsilon.$$

Example 5 Use Cauchy Criterion to show that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Proof Let $\varepsilon>0$ be given. We need to show that there exists an integer N such that for every $n>N$ and for every $p\in\mathbf{N}^*$,

$$\left|\sum_{k=n+1}^{n+p}\frac{1}{k^2}\right| = \frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+p)^2}$$
$$<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots+\frac{1}{(n+p-1)(n+p)}$$
$$=\frac{1}{n}-\frac{1}{n+p}<\frac{1}{n}<\varepsilon.$$

We may choose $N=\left[\frac{1}{\varepsilon}\right]$, then $n>N$ implies that $\frac{1}{n}<\varepsilon$ for every $p\in\mathbf{N}^*$. Thus $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

§ 5.1.3 Properties of Series

Whenever we have two convergent series, we can add, subtract them term by term, or multiply them by constants to make new convergent series.

Theorem 3 (Linearity of Convergent Series) If $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ both converge, and if c is a constant, then $\sum_{n=1}^{\infty}(u_n \pm v_n)$ and $\sum_{n=1}^{\infty}cu_n$ also converge, and

(i) $\sum_{n=1}^{\infty}(u_n \pm v_n) = \sum_{n=1}^{\infty}u_n \pm \sum_{n=1}^{\infty}v_n$;

(ii) $\sum_{n=1}^{\infty}cu_n = c\sum_{n=1}^{\infty}u_n$.

Proof (i) Let $S=\sum_{n=1}^{\infty}u_n$ and $T=\sum_{n=1}^{\infty}v_n$. The partial sums are given by $S_n=\sum_{k=1}^{n}u_k$ and $T_n=\sum_{k=1}^{n}v_k$. Then

$$\sum_{n=1}^{\infty}(u_n \pm v_n) = \lim_{n\to\infty}\sum_{k=1}^{n}(u_k \pm v_k) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}u_k \pm \sum_{k=1}^{n}v_k\right)$$
$$= \lim_{n\to\infty}(S_n \pm T_n) = \lim_{n\to\infty}S_n \pm \lim_{n\to\infty}T_n$$
$$= S \pm T = \sum_{n=1}^{\infty}u_n \pm \sum_{n=1}^{\infty}v_n.$$

(ii)
$$\sum_{n=1}^{\infty}cu_n = \lim_{n\to\infty}\sum_{k=1}^{n}cu_k = \lim_{n\to\infty}c\sum_{k=1}^{n}u_k = \lim_{n\to\infty}cS_n$$
$$= c\lim_{n\to\infty}S_n = cS = c\sum_{n=1}^{\infty}u_n.$$

As corollaries of Theorem 3, we have:

(1) Every nonzero constant multiple of a divergent series diverges;

(2) If $\sum_{n=1}^{\infty} u_n$ converges and $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} (u_n + v_n)$ and $\sum_{n=1}^{\infty} (u_n - v_n)$ both diverge.

Theorem 4 The convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$ is not affected by adding or removing a finite number of terms.

Proof Consider the series $\sum_{n=m+1}^{\infty} u_n$, where $m \in \mathbf{N}^*$. Let S_n and S'_n be the n-th partial sum of the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=m+1}^{\infty} u_n$ respectively. Then

$$\begin{aligned} S'_n &= u_{m+1} + u_{m+2} + \cdots + u_{m+n} \\ &= (u_1 + u_2 + \cdots + u_m + u_{m+1} + u_{m+2} + \cdots + u_{m+n}) - (u_1 + u_2 + \cdots + u_m) \\ &= S_{m+n} - S_m, \end{aligned}$$

where S_m is a constant.

Now suppose that $\sum_{n=1}^{\infty} u_n$ converges and $S = \sum_{n=1}^{\infty} u_n$. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{m+n} = S$. Therefore

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty}(S_{m+n} - S_m) = S - S_m.$$

Hence the series $\sum_{n=m+1}^{\infty} u_n$ converges, and its sum is $S - S_m$.

Conversely, if $\sum_{n=m+1}^{\infty} u_n$ converges, and the sum is S', then

$$\lim_{n\to\infty} S_{m+n} = \lim_{n\to\infty}(S'_n + S_m) = S' + S_m.$$

Hence the series $\sum_{n=1}^{\infty} u_n$ converges, and its sum is $S' + S_m$.

The Theorem does not say that the sum does not change if finitely many terms are changed, but merely that such a change cannot transform a convergent series into a divergent one or vice versa. Thus the series $300 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$ converges, while the series $\sum_{n=1\,000\,000}^{\infty} \frac{1}{n}$ diverges.

Theorem 5 The terms of a convergent series can be grouped in any way (provided that the order of the terms is maintained), and the new series will converge with the same sum as the original series.

Proof Let $\sum_{n=1}^{\infty} u_n$ be the original convergent series and $\{S_n\}$ be its sequence of partial sums. If $\sum_{m=1}^{\infty} v_m$ is a series formed by grouping the terms of $\sum_{n=1}^{\infty} u_n$ and if $\{T_m\}$ is its sequence of partial sums, then each T_m is one of the S_n's. For example, T_5 might be

$$T_5 = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6) + (a_7 + a_8) + a_9$$

in which case $T_5 = S_9$. Thus, $\{T_m\}$ is a subsequence of $\{S_n\}$. A moment's thought should convince you that if $S_n \to S$ then $T_m \to S$.

However, the converse of Theorem 5 is not true. For example, the series

$$1-1+1-1+\cdots+(-1)^{n-1}+\cdots$$

has partial sums

$$S_1=1, \quad S_2=0, \quad S_3=1, \quad S_4=0, \quad \cdots.$$

The sequence of partial sums, 1, 0, 1, 0, 1, ⋯, diverges; thus the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges. We might, however, group the terms in this series as follows:

$$(1-1)+(1-1)+\cdots+(1-1)+\cdots=0+0+\cdots.$$

This series converges obviously.

Example 6 Calculate $\sum_{n=1}^{\infty} \frac{4^{n+1}-3 \cdot 2^n}{5^n}$.

Solution Since the geometric series $\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$ and $\sum_{n=1}^{\infty} \frac{3 \cdot 2^n}{5^n}$ converges, by Theorem 3,

$$\sum_{n=1}^{\infty} \frac{4^{n+1}-3 \cdot 2^n}{5^n} = \frac{16}{5} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n - \frac{6}{5} \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$$
$$= \frac{16}{5} \cdot \frac{1}{1-\frac{4}{5}} - \frac{6}{5} \cdot \frac{1}{1-\frac{2}{5}} = 14.$$

Example 7 Indicate whether the given series converges or diverges.

(1) $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{(-1)^n \cdot 2^n}{3^n}\right)$; (2) $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{3}{4^n}\right)$.