Kung-Ching Chang

Springer
Monographs
in
Mathematics

Methods in Nonlinear Analysis

非线性分析方法

Springer

世界图书出版公司
www.wpcbj.com.cn

Kung-Ching Chang

Methods in Nonlinear Analysis

Springer

图书在版编目（CIP）数据

非线性分析方法 = Methods in nonlinear analysis：英文/张恭庆著．—影印本．—北京：世界图书出版公司北京公司，2014.2

ISBN 978-7-5100-7593-3

Ⅰ.①非… Ⅱ.①张… Ⅲ.①非线性—分析方法—英文 Ⅳ.①O177.91-34

中国版本图书馆 CIP 数据核字（2014）第 026500 号

书　　名：	Methods in Nonlinear Analysis
作　　者：	Kung-Ching Chang
中 译 名：	非线性分析方法
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24 开
印　　张：	19
版　　次：	2014 年 5 月
版权登记：	图字：01-2013-2382
书　　号：	978-7-5100-7593-3　　**定　　价：** 69.00 元

Preface

Nonlinear analysis is a new area that was born and has matured from abundant research developed in studying nonlinear problems. In the past thirty years, nonlinear analysis has undergone rapid growth; it has become part of the mainstream research fields in contemporary mathematical analysis.

Many nonlinear analysis problems have their roots in geometry, astronomy, fluid and elastic mechanics, physics, chemistry, biology, control theory, image processing and economics. The theories and methods in nonlinear analysis stem from many areas of mathematics: Ordinary differential equations, partial differential equations, the calculus of variations, dynamical systems, differential geometry, Lie groups, algebraic topology, linear and nonlinear functional analysis, measure theory, harmonic analysis, convex analysis, game theory, optimization theory, etc. Amidst solving these problems, many branches are intertwined, thereby advancing each other.

The author has been offering a course on nonlinear analysis to graduate students at Peking University and other universities every two or three years over the past two decades. Facing an enormous amount of material, vast numbers of references, diversities of disciplines, and tremendously different backgrounds of students in the audience, the author is always concerned with how much an individual can truly learn, internalize and benefit from a mere semester course in this subject.

The author's approach is to emphasize and to demonstrate the most fundamental principles and methods through important and interesting examples from various problems in different branches of mathematics. However, there are technical difficulties: Not only do most interesting problems require background knowledge in other branches of mathematics, but also, in order to solve these problems, many details in argument and in computation should be included. In this case, we have to get around the real problem, and deal with a simpler one, such that the application of the method is understandable. The author does not always pursue each theory in its broadest generality; instead, he stresses the motivation, the success in applications and its limitations.

The book is the result of many years of revision of the author's lecture notes. Some of the more involved sections were originally used in seminars as introductory parts of some new subjects. However, due to their importance, the materials have been reorganized and supplemented, so that they may be more valuable to the readers.

In addition, there are notes, remarks, and comments at the end of this book, where important references, recent progress and further reading are presented.

The author is indebted to Prof. Wang Zhiqiang at Utah State University, Prof. Zhang Kewei at Sussex University and Prof. Zhou Shulin at Peking University for their careful reading and valuable comments on Chaps. 3, 4 and 5.

Peking University
September, 2003

Kung Ching Chang

Contents

1

Linearization

The first and the easiest step in studying a nonlinear problem is to linearize it. That is, to approximate the initial nonlinear problem by a linear one. Nonlinear differential equations and nonlinear integral equations can be seen as nonlinear equations on certain function spaces. In dealing with their linearizations, we turn to the differential calculus in infinite-dimensional spaces. The implicit function theorem for finite-dimensional space has been proved very useful in all differential theories: Ordinary differential equations, differential geometry, differential topology, Lie groups etc. In this chapter we shall see that its infinite-dimensional version will also be useful in partial differential equations and other fields; in particular, in the local existence, in the stability, in the bifurcation, in the perturbation problem, and in the gluing technique etc. This is the contents of Sects. 1.2 and 1.3. Based on Newton iterations and the smoothing operators, the Nash–Moser iteration, which is motivated by the isometric embeddings of Riemannian manifolds into Euclidean spaces and the KAM theory, is now a very important tool in analysis. Limited in space and time, we restrict ourselves to introducing only the spirit of the method in Sect. 1.4.

1.1 Differential Calculus in Banach Spaces

There are two kinds of derivatives in the differential calculus of several variables, the gradients and the directional derivatives. We shall extend these two to infinite-dimensional spaces.

Let X, Y and Z be Banach spaces, with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, respectively. If there is no ambiguity, we omit the subscripts. Let $U \subset X$ be an open set, and let $f : U \to Y$ be a map.

1.1.1 Frechet Derivatives and Gateaux Derivatives

Definition 1.1.1 *(Fréchet derivative) Let $x_0 \in U$; we say that f is Fréchet differentiable (or F-differentiable) at x_0, if $\exists A \subset L(X,Y)$ such that*

$$\| f(x) - f(x_0) - A(x - x_0) \|_Y = \circ(\| x - x_0 \|_X) .$$

Let $f'(x_0) = A$, and call it the Fréchet (or F-) derivative of f at x_0.

If f is F-differentiable at every point in U, and if $x \mapsto f'(x)$, as a mapping from U to $L(X,Y)$, is continuous at x_0, then we say that f is continuously differentiable at x_0. If f is continuously differentiable at each point in U, then we say that f is continuously differentiable on U, and denote it by $f \in C^1(U,Y)$.

Parallel to the differential calculus of several variables, by definition, we may prove the following:

1. If f is F-differentiable at x_0, then $f'(x_0)$ is uniquely determined.
2. If f is F-differentiable at x_0, then f must be continuous at x_0.
3. (Chain rule) Assume that $U \subset X, V \subset Y$ are open sets, and that f is F-differentiable at x_0, and g is F-differentiable at $f(x_0)$, where

$$U \xrightarrow{\ f\ } V \xrightarrow{\ g\ } Z$$

Then

$$(g \circ f)'(x_0) = g' \circ f(x_0) \cdot f'(x_0) .$$

Definition 1.1.2 *(Gateaux derivative) Let $x_0 \in U$; we say that f is Gateaux differentiable (or G-differentiable) at x_0, if $\forall h \in X, \exists\, df(x_0,h) \subset Y$, such that*

$$\|f(x_0 + th) - f(x_0) - tdf(x_0,h)\|_Y = \circ(t) \quad as \quad t \to 0$$

for all $x_0 + th \subset U$. We call $df(x_0,h)$ the Gateaux derivative (or G-derivative) of f at x_0.

We have

$$\frac{d}{dt} f(x_0 + th) \mid_{t=0} = df(x_0,h) ,$$

if f is G-differentiable at x_0.

By definition, we have the following properties:

1. If f is G-differentiable at x_0, then $df(x_0,h)$ is uniquely determined.
2. $df(x_0,th) = tdf(x_0,h) \quad \forall t \in \mathbb{R}^1$.
3. If f is G-differentiable at x_0, then $\forall h \in X$, $\forall y^* \in Y^*$, the function $\varphi(t) = \langle y^*, f(x_0 + th)\rangle$ is differentiable at $t = 0$, and $\varphi'(t) = \langle y^*, df(x_0,h)\rangle$.
4. Assume that $f : U \to Y$ is G-differentiable at each point in U, and that the segment $\{x_0 + th \mid t \in [0,1]\} \subset U$, then

$$\| f(x_0 + h) - f(x_0) \|_Y \leqslant \sup_{0<t<1} \| df(x_0 + th, h) \|_Y$$

Proof. Let

$$\varphi_{y^*}(t) = \langle y^*, f(x_0 + th) \rangle \quad t \in [0,1], \forall y^* \in Y^*$$

$$\begin{aligned} |\langle y^*, f(x_0+h) - f(x_0) \rangle| &= |\varphi_{y^*}(1) - \varphi_{y^*}(0)| \\ &= |\varphi'_{y^*}(t^*)| \\ &= |\langle y^*, df(x_0 + t^* h, h) \rangle| \end{aligned}$$

for some $t^* \in (0,1)$ depending on y^*. The conclusion follows from the Hahn–Banach theorem. □

5. If f is F-differentiable at x_0, then f is G-differentiable at x_0, with $df(x_0, h) = f'(x_0)h \quad \forall h \in X$.

Conversely it is not true, but we have:

Theorem 1.1.3 *Suppose that $f : U \to Y$ is G-differentiable, and that $\forall x \in U$, $\exists A(x) \in L(X,Y)$ satisfying*

$$df(x,h) = A(x)h \qquad \forall h \in X\ .$$

If the mapping $x \mapsto A(x)$ is continuous at x_0, then f is F-differentiable at x_0, with $f'(x_0) = A(x_0)$.

Proof. With no loss of generality, we assume that the segment $\{x_0 + th \mid t \in [0,1]\}$ is in U. According to the Hahn–Banach theorem, $\exists y^* \in Y^*$, with $\| y^* \| = 1$, such that

$$\| f(x_0+h) - f(x_0) - A(x_0)h \|_Y = \langle y^*, f(x_0+h) - f(x_0) - A(x_0)h \rangle\ .$$

Let

$$\varphi(t) = \langle y^*, f(x_0 + th) \rangle\ .$$

From the mean value theorem, $\exists \xi \in (0,1)$ such that

$$\begin{aligned} |\varphi(1) - \varphi(0) - \langle y^*, A(x_0)h \rangle| &= |\varphi'(\xi) - \langle y^*, A(x_0)h \rangle| \\ &= |\langle y^*, df(x_0 + \xi h, h) - A(x_0)h) \rangle| \\ &= |\langle y^*, [A(x_0 + \xi h) - A(x_0)]h \rangle| \\ &= \circ(\| h \|)\ , \end{aligned}$$

i.e., $f'(x_0) = A(x_0)$. □

The importance of Theorem 1.1.3 lies in the fact that it is not easy to write down the F-derivative for a given map directly, but the computation of G-derivative is reduced to the differential calculus of single variables. The same situation occurs in the differential calculus of several variables: Gradients are reduced to partial derivatives, and partial derivatives are reduced to derivatives of single variables.

Example 1. Let $A \in L(X,Y)$, $f(x) = Ax$. Then $f'(x) = A \quad \forall x$.

Example 2. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and let $\varphi_1, \varphi_2 \dots, \varphi_m \in C^1(\mathbb{R}^n, R^1)$. Set

$$f(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_m(x) \end{pmatrix}, \text{ i.e.,} \quad f : X \to Y .$$

Then

$$f'(x_0) = \left(\frac{\partial \varphi_i(x_0)}{\partial x_j} \right)_{m \times n} .$$

Example 3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Denote by $C(\overline{\Omega})$ the continuous function space on $\overline{\Omega}$. Let

$$\varphi : \overline{\Omega} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1 ,$$

be a C^1 function. Define a mapping $f : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$u(x) \mapsto \varphi(x, u(x)) .$$

Then f is F-differentiable, and $\forall u_0 \in C(\overline{\Omega})$,

$$(f'(u_0) \cdot v)(x) = \varphi_u(x, u_0(x)) \cdot v(x) \quad \forall v \in C(\overline{\Omega}) .$$

Proof. $\forall h \in C(\overline{\Omega})$

$$t^{-1}[f(u_0 + th) - f(u_0)](x) = \varphi_u(x, u_0(x) + t\theta(x)h(x))h(x) ,$$

where $\theta(x) \in (0,1)$. $\forall \varepsilon > 0$, $\forall M > 0$, $\exists \delta = \delta(M, \varepsilon) > 0$ such that

$$| \varphi_u(x,\xi) - \varphi_u(x,\xi') | < \varepsilon, \ \forall x \in \overline{\Omega} ,$$

as $|\xi|$, $|\xi'| \leqslant M$ and $|\xi - \xi'| \leqslant \delta$. We choose $M = \| u_0 \| + \| h \|$, then for $|t| < \delta < 1$,

$$|\varphi_u(x, u_0(x) + t\theta(x)h(x)) - \varphi_u(x, u_0(x))| < \varepsilon .$$

It follows that $df(u_0, h)(x) = \varphi_u(x, u_0(x))h(x)$.

Noticing that the multiplication operator $h \mapsto A(u)h = \varphi_u(x, u(x)) \cdot h(x)$ is linear and continuous, and the mapping $u \mapsto A(u)$ from $C(\overline{\Omega})$ into $L(C(\overline{\Omega}), C(\overline{\Omega}))$ is continuous, from Theorem 1.1.3, f is F-differentiable, and

$$(f'(u_0) \cdot v)(x) = \varphi_u(x, u_0(x)) \cdot v(x) \ \forall v \in C(\overline{\Omega}) .$$

□

We investigate nonlinear differential operators on more general spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let m be a nonnegative integer, $\gamma \in (0,1)$. $C^m(\overline{\Omega})$ (and the Hölder space $C^{m,\gamma}(\overline{\Omega})$) is defined to be the function space consisting of C^m functions (with γ-Hölder continuous m-order partial derivatives).

The norms are defined as follows:

$$\| u \|_{C^m} = \max_{x \in \overline{\Omega}} \sum_{|\alpha| \le m} |\partial^\alpha u(x)| ,$$

and

$$\| u \|_{C^{m,\gamma}} = \| u \|_{C^m} + \max_{x,y \in \overline{\Omega}\ |\alpha| = m} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma} ,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$.

We always denote by m^* the number of the index set $\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid |\alpha| \leqslant m\}$, and $D^m u$ the set $\{\partial^\alpha u \mid |\alpha| \leqslant m\}$.

Suppose that r is a nonnegative integer, and that $\varphi \in C^\infty(\overline{\Omega} \times \mathbb{R}^{r^*})$. Define a differentiable operator of order r:

$$f(u)(x) = \varphi(x, D^r u(x)) .$$

Suppose $m \geqslant r$, then $f : C^m(\overline{\Omega}) \to C^{m-r}(\overline{\Omega})$ (and also $C^{m,\gamma}(\overline{\Omega}) \to C^{m-r,\gamma}(\overline{\Omega})$) is F-differentiable. Furthermore

$$(f'(u_0)h)(x) = \sum_{|\alpha| \le r} \varphi_\alpha(x, D^r u_0(x)) \cdot \partial^\alpha h(x), \quad \forall h \in C^m(\overline{\Omega}) ,$$

where φ_α is the partial derivative of φ with respect to the variable index α.

The proof is similar to Example 3.

Example 4. Suppose $\varphi \in C^\infty(\overline{\Omega} \times \mathbb{R}^{r^*})$. Define

$$f(u) = \int_\Omega \varphi(x, D^r u(x)) dx \quad \forall u \in C^r(\overline{\Omega}) .$$

Then $f : C^r(\overline{\Omega}) \to \mathbb{R}^1$ is F-differentiable. Furthermore

$$\langle f'(u_0), h \rangle = \int_\Omega \sum_{|\alpha| \le r} \varphi_\alpha(x, D^r u_0(x)) \partial^\alpha h(x) dx \quad \forall h \in C^r(\overline{\Omega}) .$$

Proof. Use the chain rule:

$$C^r(\overline{\Omega}) \xrightarrow{\varphi(\cdot, D^r u(\cdot))} C(\overline{\Omega}) \xrightarrow{\int_\Omega} \mathbb{R}^1 ,$$

and combine the results of Examples 1 and 3. □

In particular, the following functional occurs frequently in the calculus of variations ($r = 1, r^* = n+1$). Assume that $\varphi(x, u, p)$ is a function of the form:

$$\varphi(x, u, p) = \frac{1}{2}|p|^2 + \sum_{i=1}^{n} a_i(x)p_i + a_0(x)u ,$$

where $p = (p_1, p_2, \ldots, p_n)$, and $a_i(x)$, $i = 0, 1, \ldots, n$, are in $C(\overline{\Omega})$.
Set

$$f(u) = \int_\Omega \left[\frac{1}{2}|\nabla u(x)|^2 + \sum_{i=1}^{n} a_i(x)\partial_{x_i}u + a_0(x)u(x)\right] dx ,$$

we have

$$\langle f'(u), h\rangle = \int_\Omega \left[\nabla u(x) \cdot \nabla h(x) + \sum_{i=1}^{n} a_i(x)\partial_{x_i}h(x) + a_0(x)h(x)\right] dx \quad \forall h \in C^1(\overline{\Omega}) .$$

Example 5. Let X be a Hilbert space, with inner product $(\,,)$. Find the F-derivative of the norm $f(x) = \| x \|$, as $x \neq \theta$.

Let $F(x) = \| x \|^2$. Since

$$t^{-1}(\| x + th \|^2 - \| x \|^2) = 2(x, h) + t \| h \|^2 ,$$

we have $dF(x, h) = 2(x, h)$. It is continuous for all x, therefore F is F-differentiable, and

$$F'(x)h = 2(x, h) .$$

Since $f = F^{\frac{1}{2}}$, by the chain rule

$$F'(x) = 2 \| x \| \cdot f'(x) .$$

As $x \neq \theta$,

$$f'(x)h = \left(\frac{x}{\| x \|}, h\right) .$$

In the applications to PDE as well as to the calculus of variations, Sobolev spaces are frequently used. We should extend the above studies to nonlinear operators defined on Sobolev spaces.

$\forall p \geqslant 1$, $\forall$ nonnegative integer m, let

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \mid |\alpha| \leqslant m\} ,$$

where $\partial^\alpha u$ stands for the α-order generalized derivative of u, i.e., the derivative in the distribution sense. Define the norm

$$\| u \|_{W^{m,p}} = \left(\sum_{|\alpha \leq m} \| \partial^\alpha u \|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} .$$

The Banach space is called the Sobolev space of index $\{m, p\}$.

$W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, and the closure of $C_0^\infty(\Omega)$ under this norm is denoted by $H_0^m(\Omega)$.

1.1.2 Nemytscki Operator

On Sobolev spaces, we extend the composition operator $u \mapsto \varphi(x, u(x))$ such that φ may not be continuous in x. The class of operators is sometimes called Nemytski operators.

Definition 1.1.4 *Let (Ω, B, μ) be a measure space. We say that $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^1$ is a Caratheodory function, if*

1. *$\forall a.e. x \in \Omega$, $\xi \mapsto \varphi(x, \xi)$ is continuous.*
2. *$\forall \xi \in \mathbb{R}^N$, $x \mapsto \varphi(x, \xi)$ is μ-measurable.*

The motivation in introducing the Caratheodory function is to make the composition function measurable if $u(x)$ is only measurable. Indeed, there exists a sequence of simple functions $\{u_n(x)\}^\infty$, such that $u_n(x) \to u(x)$ *a.e.*, $\varphi(x, u_n(x))$ is measurable according to (2). And from (1), $\varphi(x, u_n(x)) \to \varphi(x, u(x))$ *a.e.*, therefore $\varphi(x, u(x))$ is measurable.

Theorem 1.1.5 *Assume $p_1, p_2 \geqslant 1$, $a > 0$ and $b \in L^{p_2}_{d\mu}(\Omega)$. Suppose that φ is a Caratheodory function satisfying*

$$|\varphi(x, \xi)| \leqslant b(x) + a|\xi|^{\frac{p_1}{p_2}} .$$

Then $f : u(x) \mapsto \varphi(x, u(x))$ is a bounded and continuous mapping from $L^{p_1}_{d\mu}(\Omega, \mathbb{R}^N)$ to $L^{p_2}_{d\mu}(\Omega, \mathbb{R}^N)$.

Proof. The boundedness follows from the Minkowski inequality:

$$\| f(u) \|_{p_2} \leqslant \| b \|_{p_2} + a \| u \|_{p_1}^{\frac{p_1}{p_2}} ,$$

where $\| \cdot \|_p$ is the $L^p_{d\mu}(\Omega, \mathbb{R}^N)$ norm. We turn to proving the continuity. It is sufficient to prove that $\forall \{u_n\}_1^\infty$ if $u_n \to u$ in L^{p_1}, then there is a subsequence $\{u_{n_i}\}$ such that $f(u_{n_i}) \to f(u)$ in L^{p_2}. Indeed one can find a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges a.e. to u, along which $\| u_{n_i} - u_{n_{i-1}} \|_{p_1} < \frac{1}{2^i}$, $i = 2, 3, \ldots$; therefore

$$|u_{n_i}(x)| \leqslant \Phi(x) := |u_{n_1}(x)| + \sum_{i=2}^{\infty} |u_{n_i}(x) - u_{n_{i-1}}(x)| .$$

Since Φ is measurable, and

$$\left(\int_\Omega |\Phi(x)|^{p_1} d\mu \right)^{\frac{1}{p_1}} \leqslant \| u_{n_1} \|_{p_1} + \sum_{i=2}^{\infty} \| u_{n_i} - u_{n_{i-1}} \|_{p_1} < +\infty ,$$

we conclude that $\Phi \in L^{p_1}_{d\mu}(\Omega)$. Noticing

$$f(u_{n_i}) = \varphi(x, u_{n_i}(x)) \to \varphi(x, u(x)) \quad a.e.\,,$$

and

$$|f(u_{n_i})(x)| \leqslant b(x) + a(\Phi(x))^{\frac{p_1}{p_2}} \in L^{p_2}_{d\mu}(\Omega, R^N)\,,$$

we have $\| f(u_{n_i}) - f(u) \|_{p_2} \to 0$, according to Lebesgue dominance theorem. This proves the continuity of f. □

Corollary 1.1.6 *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and let $1 \leq p_1, p_2 \leq \infty$. Suppose that $\varphi : \Omega \times \mathbb{R}^{m^*} \to \mathbb{R}$ is a Caratheodory function satisfying*

$$|\varphi(x, \xi_0, \dots, \xi_m)| \leqslant b(x) + a \sum_{j=0}^{m} |\xi_j|^{\frac{\alpha_j}{p_2}}\,,$$

where ξ_j is a $\#\{\alpha = (\alpha_1, \dots, \alpha_n) \mid |\alpha| = j\}$-vector, $\alpha_j \leqslant (\frac{1}{p_1} - \frac{m-j}{n})^{-1}$, $a > 0$, and $b \in L^{p_2}(\Omega)$. Then $f(u)(x) = \varphi(x, D^m u(x))$ defines a bounded and continuous map from $W^{m,p_1}(\Omega)$ into $L^{p_2}(\Omega)$.

Corollary 1.1.7 *Suppose that $\Omega \subset \mathbb{R}^n$ and that $\varphi : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ and $\varphi_\xi(x, \xi)$ are Caratheodory functions. If $|\varphi_\xi(x, \xi)| \leqslant b(x) + a|\xi|^r$, where $b \in L^{\frac{2n}{n+2}}(\Omega)$, $a > 0$, and $r = \frac{n+2}{n-2}$ (if $n \leq 2$, then the restriction is not necessary), then the functional*

$$f(u) = \int_\Omega \varphi(x, u(x))dx$$

is F-differentiable on $H^1(\Omega)$, with F-derivative

$$\langle f'(u), v \rangle = \int_\Omega \varphi_\xi(x, u(x)) \cdot v(x)dx\,,$$

where $\langle , \rangle$ is the inner product on $H^1(\Omega)$.

Proof. The Sobolev embedding theorem says that the injection $i : H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is continuous, so is the dual map $i^* : L^{\frac{2n}{n+2}}(\Omega) \hookrightarrow (H^1(\Omega))^*$.

According to Theorem 1.1.5, $\varphi_\xi(\cdot, \cdot) : L^{\frac{2n}{n-2}} \to L^{\frac{2n}{n+2}}$ is continuous. Therefore the Gateaux derivative

$$df(u, v) = \int_\Omega \varphi_\xi(x, u(x)) \cdot v(x)dx \quad \forall v \in H^1(\Omega)$$

is continuous from $H^1(\Omega)$ to $(H^1(\Omega))^*$. Applying Theorem 1.1.3, we conclude that f is F-differentiable on $H^1(\Omega)$. The proof is complete. □

Corollary 1.1.8 *In Corollary 1.1.6, the differential operator*

$$f(u(x)) = \varphi(x, D^m u(x))$$

from $C^{l,\gamma}(\overline{\Omega})$ to $C^{l-m,\gamma}(\overline{\Omega})$, $l \geqslant m$, $0 \leqslant \gamma < 1$, is F-differentiable, with

$$(f'(u_0)h)(x) = \sum_{|\alpha| \leq m} \varphi_\alpha(x, D^m u(x))\partial^\alpha h(x) \quad \forall h \in C^{l,\gamma}(\overline{\Omega})\,.$$

1.1.3 High-Order Derivatives

The second-order derivative of f at x_0 is defined to be the derivative of $f'(x)$ at x_0. Since $f' : U \to L(X, Y)$, $f''(x_0)$ should be in $L(X, L(X, Y))$. However, if we identify the space of bounded bilinear mappings with $L(X, L(X, Y))$, and verify that $f''(x_0)$ as a bilinear mapping is symmetric, see Theorem 1.1.9 below, then we can define equivalently the second derivative $f''(x_0)$ as follows: For $f : U \to Y$, $x_0 \in U \subset X$, if there exists a bilinear mapping $f''(x_0)(\cdot, \cdot)$ of $X \times X \to Y$ satisfying

$$\| f(x_0+h)-f(x_0)-f'(x_0)h-\frac{1}{2}f''(x_0)(h,h) \|= \circ(\| h \|^2) \quad \forall h \in X, \text{as } \| h \| \to 0 ,$$

then $f''(x_0)$ is called the second-order derivative of f at x_0.

By the same manner, one defines the mth-order derivatives at x_0 successively: $f^{(m)}(x_0) : X \times \cdots \times X \to Y$ is an m-linear mapping satisfying

$$\left\| f(x_0 + h) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(h, \ldots, h)}{j!} \right\| = \circ(\| h \|^m) ,$$

as $\| h \| \to 0$. Then f is called m differentiable at x_0.

Similar to the finite-dimensional vector functions, we have:

Theorem 1.1.9 *Assume that $f : U \to Y$ is m differentiable at $x_0 \in U$. Then for any permutation π of $(1, \ldots, m)$, we have*

$$f^{(m)}(x_0)(h_1, \ldots, h_m) = f^{(m)}(x_0)(h_{\pi(1)}, \ldots, h_{\pi(m)}) .$$

Proof. We only prove this in the case where $m = 2$, i.e.,

$$f''(x_0)(\xi, \eta) = f''(x_0)(\eta, \xi) \quad \forall \xi, \eta \in X .$$

Indeed $\forall y^* \in Y^*$, we consider the function

$$\varphi(t, s) = \langle y^*, f(x_0 + t\xi + s\eta)\rangle .$$

It is twice differentiable at $t = s = 0$; so is

$$\frac{\partial^2}{\partial t \partial s}\varphi(0,0) = \frac{\partial^2}{\partial s \partial t}\varphi(0,0) .$$

Since $f'(x_0 + t\xi + s\eta)$ is continuous as $|t|$, $|s|$ small, one has

$$\frac{\partial}{\partial s}\varphi(t, \cdot)|_{s=0} = \langle y^*, f'(x_0 + t\xi)\eta\rangle ;$$

and then,

$$\frac{\partial^2}{\partial t \partial s}\varphi(t, s)|_{t=s=0} = \langle y^*, f''(x_0)(\xi, \eta)\rangle .$$

Similarly

$$\frac{\partial^2}{\partial s \partial t}\varphi(t, s)|_{t=s=0} = \langle y^*, f''(x_0)(\eta, \xi)\rangle .$$

This proves the conclusion. □