ALM 26

Advanced Lectures in Mathematics

Handbook of Moduli

(Volume III)

模手册(卷Ⅲ)

Editors: Gavril Farkas · Ian Morrison

高等教育出版社
HIGHER EDUCATION PRESS

ALM 26
Advanced Lectures in Mathematics

Handbook of Moduli

(Volume III)

模手册（卷 III） Moshouce

Editors: Gavril Farkas · Ian Morrison

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

International Press

图书在版编目(CIP)数据

模手册. 3 = Handbook of Moduli. Vol. Ⅲ : 英文 / (德)法卡斯(Farkas, G.),(美)莫里森(Morrison, I.)编. — 北京 : 高等教育出版社, 2013.1
ISBN 978-7-04-035174-3

Ⅰ.①模… Ⅱ.①法… ②莫… Ⅲ.①代数几何-文集-英文 Ⅳ.①O187-53

中国版本图书馆CIP数据核字(2012)第242457号

策划编辑 王丽萍　　责任编辑 王丽萍　　封面设计 张申申　　责任印制 朱学忠

出版发行	高等教育出版社	咨询电话	400-810-0598
社　　址	北京市西城区德外大街4号	网　　址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印　　刷	涿州市星河印刷有限公司	网上订购	http://www.landraco.com
开　　本	787mm × 1092mm　1/16		http://www.landraco.com.cn
印　　张	37.25	版　　次	2013年1月第1版
字　　数	710千字	印　　次	2013年1月第1次印刷
购书热线	010-58581118	定　　价	128.00元

物 料 号　35174-00

ADVANCED LECTURES IN MATHEMATICS

ADVANCED LECTURES IN MATHEMATICS

(*Executive Editors: Shing-Tung Yau, Kefeng Liu, Lizhen Ji*)

1. Superstring Theory (2007)
(Editors: Shing-Tung Yau, Kefeng Liu, Chongyuan Zhu)

2. Asymptotic Theory in Probability and Statistics with Applications (2007)
(Editors: Tze Leung Lai, Lianfen Qian, Qi-Man Shao)

3. Computational Conformal Geometry (2007)
(Authors: Xianfeng David Gu, Shing-Tung Yau)

4. Variational Principles for Discrete Surfaces (2007)
(Authors: Feng Luo, Xianfeng David Gu, Junfei Dai)

5. Proceedings of The 4th International Congress of Chinese Mathematicians Vol. I, II (2007)
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)

6. Geometry, Analysis and Topology of Discrete Groups (2008)
(Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau)

7. Handbook of Geometric Analysis Vol. I (2008)
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)

8. Recent Developments in Algebra and Related Areas (2009)
(Editors: Chongying Dong, Fu-An Li)

9. Automorphic Forms and the Langlands Program (2009)
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)

10. Trends in Partial Differential Equations (2009)
(Editors: Baojun Bian, Shenghong Li, Xu-Jia Wang)

11. Recent Advances in Geometric Analysis (2009)
(Editors: Yng-Ing Lee, Chang-Shou Lin, Mao-Pei Tsui)

12. Cohomology of Groups and Algebraic K-theory (2009)
(Editors: Lizhen Ji, Kefeng Liu, Shing-Tung Yau)

13–14. Handbook of Geometric Analysis Vol. I, II (2010)
(Editors: Lizhen Ji, Peter Li, Richard Schoen, Leon Simon)

15. An Introduction to Groups and Lattices (2010)
(Author: Robert L. Griess, Jr.)

16. Transformation Groups and Moduli Spaces of Curves (2010)
(Editors: Lizhen Ji, Shing-Tung Yau)

17–18. Geometry and Analysis Vol. I, II (2010)
(Editor: Lizhen Ji)

19. Arithmetic Geometry and Automorphic Forms (2011)
(Editors: James Cogdell, Jens Funke, Michael Rapoport, Tonghai Yang)

20. Surveys in Geometric Analysis and Relativity (2011)
(Editors: Hubert L. Bray, William P. Minicozzi II)

21. Advances in Geometric Analysis (2011)
(Editors: Stanisław Janeczko, Jun Li, Duong H. Phong)

22. Differential Geometry (2012)
(Editors: Yibing Shen, Zhongmin Shen, Shing-Tung Yau)

23. Recent Development in Geometry and Analysis (2012)
(Editors: Yuxin Dong, Jixiang Fu, Guozhen Lu, Weimin Sheng, Xiaohua Zhu)

24–26. Handbook of Moduli Vol. I, II, III (2012)
(Editors: Gavril Farkas, Ian Morrison)

Higher Education Press Limited Company
4 Dewai Dajie, Beijing 100120, P. R. China, and
International Press
387 Somerville Ave, Somerville, MA, U. S. A.

The Handbook of Moduli was designed by Ian Morrison in LaTeX using a variant of the standard style files of the Higher Education Press. The text of the Handbook is set in ITC Giovanni and the mathematics in AMS Euler.

Giovanni was designed by Robert Slimbach in 1989 for ITC and was one of the early faces that earned him the Prix Charles Peignot, the Fields Medal of type design awarded "to a designer under the age of 35 who has made an outstanding contribution to type design". It combines the basic proportions of traditional oldstyle designs with the more even color and higher x-height of modern digital fonts to produce an inconspicuous but legible typeface.

Euler was designed in 1981 by Hermann Zapf, a major figure in 20th type design and a pioneer in digital typography, working in close cooperation with Donald Knuth, as an upright, cursive symbol font that would give the effect of mathematics handwritten on a blackboard. In 2008, Zapf reshaped many of the glyphs, with the assistance of Hans Hagen, Taco Hoekwater, and Volker RW Schaa, in order to harmonize the designs and bring them into line with contemporary standards of digital typography.

ADVANCED LECTURES IN MATHEMATICS

The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

Contents

Volume I

Volume II

Volume III

Birational geometry for nilpotent orbits

Yoshinori Namikawa

Abstract. The following topics are discussed:

(1) Basic facts and examples of resolutions for nilpotent orbit.

(2) $\mathbb{Q}$-factorial terminalizations of nilpotent orbit closures and related birational geometry.

(3) Poisson deformations of nilpotent orbit closures.

Contents

2010 *Mathematics Subject Classification.* Primary 14E30, 17B08, 17B63; Secondary 14J17, 14L17.

Key words and phrases. Birational geometry, Nilpotent orbits, Poisson deformations.

1. Introduction

The aim of this paper is to give an account of the birational point of view on nilpotent orbits in a complex simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra and G the adjoint group. An adjoint orbit O in $\mathfrak{g}$ is called a nilpotent orbit if O consists of nilpotent elements of $\mathfrak{g}$. The closure $\bar{O}$ of O is then an affine variety with singularities. In general, $\bar{O}$ is not necessarily normal (see for example [15] in this direction). In this paper we shall take its normalization $\tilde{O}$ of $\bar{O}$ and consider the birational geometry on its (partial) resolutions. Each variety $\tilde{O}$ has symplectic singularities. More precisely, the smooth locus $\tilde{O}_{reg}$ admits the Kostant-Kirillov 2-form ω, which is d-closed and non-degenerate. Moreover, if we take a resolution $\mu : Y \to \tilde{O}$, then ω extends to a regular 2-form on Y. A resolution $\mu : Y \to \tilde{O}$ is called a crepant resolution if $K_Y = \mu^* K_{\tilde{O}}$. The nilpotent cone N is defined to be the subset of $\mathfrak{g}$ which consists of all nilpotent elements of $\mathfrak{g}$. By definition N is a disjoint union of all nilpotent orbits of $\mathfrak{g}$. There is a largest nilpotent orbit O_r and N coincides with its closure. Moreover, N is a normal variety. Let B be a Borel subgroup of G and let $T^*(G/B)$ be the cotangent bundle of the flag variety G/B. By using the Killing form of $\mathfrak{g}$, one can identify $T^*(G/B)$ with a vector bundle $G \times^B [\mathfrak{b}, \mathfrak{b}]$ over G/B. Then there is a natural map

$$\nu : G \times^B [\mathfrak{b}, \mathfrak{b}] \to \mathfrak{g}$$

defined by $[g, x] \to Ad_g(x)$. The image of ν coincides with N and ν gives a resolution of N ([25]). We call ν the *Springer resolution* of N. Since $T^*(G/B)$ admits a canonical symplectic 2-form and it coincides with the pull-back of the Kostant-Kirillov 2-form on O_r, the Springer resolution is a crepant resolution. One can generalize this construction to a parabolic subgroup Q of G. Let us start with the cotangent bundle $T^*(G/Q)$. Note that $T^*(G/Q)$ is identified with $G \times^Q \mathfrak{n}(\mathfrak{q})$ where $\mathfrak{n}(\mathfrak{q})$ is the nil-radical of $\mathfrak{q}$. In a similar way to the above, we have a map

$$\nu : T^*(G/Q) \to \mathfrak{g},$$

whose image is the closure of a nilpotent orbit O. In general, ν is not birational onto its image, but a generically finite projective morphism (see 2.6 for a non-birational Springer map). When ν gives a resolution of $\bar{O}$, we call ν the Springer resolution of $\bar{O}$. In this case, the Stein factorization

$$T^*(G/Q) \overset{\nu^n}{\to} \tilde{O} \to \bar{O}$$

gives a crepant resolution of $\tilde{O}$. B. Fu [7] proved the following.

Theorem ([7]). *Let O be a nilpotent orbit of $\mathfrak{g}$ and assume that $\tilde{O}$ admits a crepant resolution. Then it coincides with a Springer resolution. More exactly, there is a parabolic subgroup Q of G such that ν^n is the given crepant resolution.*

However there still remain interesting problems. At first, there actually exists a nilpotent orbit which has no crepant resolutions. Secondly, if $\tilde{O}$ has a crepant resolution, it is not unique, that is, the choice of Q is not unique even up to conjugacy class. Our purpose is to survey complete answers (cf. [18], [19] , [21] and [8]) to these problems.

A substitute for a crepant resolution is a *$\mathbb{Q}$-factorial terminalization*. A birational projective morphism $\mu : Y \to \tilde{O}$ is a $\mathbb{Q}$-factorial terminalization if Y has only $\mathbb{Q}$-factorial terminal singularities and $K_Y = \mu^* K_{\tilde{O}}$. The existence of a $\mathbb{Q}$-factorial terminalization is established by Birkar, Cascini, Hacon and M^{c}Kernan [2]. But, we shall give here more concrete forms of $\mathbb{Q}$-factorial terminalization. A hint is already in the work of Lusztig and Spaltenstein [17]. They introduced the notion of an *induced orbit*. Let us start with a parabolic subgroup Q of G and its Levi factor L(Q). Let $O' \subset \mathfrak{l}(\mathfrak{q})$ be a nilpotent orbit with respect to the adjoint L(Q)-action. Then one can make an associated bundle $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}')$ and define a map

$$\nu : G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}') \to \mathfrak{g}$$

by $\nu([g, x]) = Ad_g(x)$. Since this is a G-equivariant closed map, its image is the closure of a nilpotent orbit O of $\mathfrak{g}$. Then we say that O is induced from O' and write $O = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}(\mathfrak{q})}(O')$. The map ν is called the generalized Springer map. The generalized Springer map ν is a generically finite projective morphism. But if ν is birational onto its image, then the Stein factorization

$$G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}') \overset{\nu^n}{\to} \tilde{O} \to \bar{O}$$

gives a partial resolution of $\tilde{O}$. Now one can prove:

Theorem 2.6. *Let O be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Then there are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ and a nilpotent orbit O' of $\mathfrak{l}(\mathfrak{q})$ such that the following holds:*

(1) $O = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}(\mathfrak{q})}(O')$.

(2) ν^n gives a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$.

In order to look for other $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$, we introduce a flat deformation of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. For simplicity we put $\mathfrak{l} := \mathfrak{l}(\mathfrak{q})$ and let L be the corresponding Levi subgroup. Let $\mathfrak{r}(\mathfrak{q})$ be the solvable radical of $\mathfrak{q}$ and consider the variety $G \times^Q (\mathfrak{r}(\mathfrak{q}) + \bar{O}')$. Its normalization $X_{\mathfrak{q},O'}$ is isomorphic to $G \times^Q (\mathfrak{r}(\mathfrak{q}) + \tilde{O}')$. Let $\mathfrak{k}$ be the center of $\mathfrak{l}$. In 3.3 we shall define a map

$$X_{\mathfrak{q},O'} \to \mathfrak{k}$$

whose central fiber $X_{\mathfrak{q},O',0}$ is $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. This map factorizes as

$$X_{\mathfrak{q},O'} \overset{\mu_{\mathfrak{q}}}{\to} \mathrm{Spec}\, \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}) \to \mathfrak{k}.$$

Put

$$Y_{\mathfrak{l},O'} := \mathrm{Spec}\, \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}).$$

An important fact is that $Y_{\mathfrak{l},O'}$ depends only on $\mathfrak{l}$ and O'. Moreover its central fiber $Y_{\mathfrak{l},O',0}$ is isomorphic to $\tilde{O}$. Define

$$\mathcal{S}(\mathfrak{l}) := \{\text{parabolic subalgebras } \mathfrak{q}' \text{ of } \mathfrak{g};\ \mathfrak{l}(\mathfrak{q}') = \mathfrak{l}\}.$$

We can define $X_{q',O'}$ for each $q' \in \mathcal{S}(\mathfrak{l})$. We also have a map

$$\mu_{q'} : X_{q',O'} \to Y_{\mathfrak{l},O'}.$$

The map $\mu_{q'}$ is a crepant birational morphism. Moreover, $\mu_{q',t}$ is an isomorphism for $t \in \mathfrak{k}^{reg}$; hence $\mu_{q'}$ is an isomorphism in codimension one. Define

$$M(L) := \mathrm{Hom}_{alg.gp}(L, \mathbb{C}^*)$$

and put $M(L)_{\mathbf{R}} := M(L) \otimes \mathbb{R}$. Then 2-nd cohomology groups $H^2(X_{q',O'}, \mathbb{R})$ are naturally identified with $M(L)_{\mathbf{R}}$. By these identifications the nef cones $\overline{\mathrm{Amp}}(\mu_{q',O'})$ are regarded as the cones in $M(L)_{\mathbf{R}}$. This leads to:

Theorem 3.14. *For $q' \in \mathcal{S}(\mathfrak{l})$, the birational map $\mu_{q'} : X_{q',O'} \to Y_{\mathfrak{l},O'}$ is a $\mathbb{Q}$-factorial terminalization and is an isomorphism in codimension one. Any $\mathbb{Q}$-factorial terminalization of $Y_{\mathfrak{l},O'}$ is obtained in this way. If $q_1 \neq q_2$, then μ_{q_1} and μ_{q_2} give different $\mathbb{Q}$-factorial terminalizations. Moreover,*

$$M(L)_{\mathbf{R}} = \cup_{q' \in \mathcal{S}(\mathfrak{l})} \overline{\mathrm{Amp}}(\mu_{q'}).$$

Two elements of $\mathcal{S}(\mathfrak{l})$ are connected by a sequence of the operations called *twists* (cf. 3.2). Corresponding to a twist $q_1 \rightsquigarrow q_2$, we have a flop

$$X_{q_1,O'} \to Z \leftarrow X_{q_2,O'}.$$

So any two $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$ are connected by a sequence of certain flops. Now let us look at the central fibers $X_{q',O',0}$ of $X_{q',O'} \to \mathfrak{k}$. The diagram

$$X_{q_1,O',0} \to Z_0 \leftarrow X_{q_2,O',0}$$

is not necessarily a flop. Twists are divided into those of the first kind and those of the second kind. If the twist $q_1 \rightsquigarrow q_2$ is of the first kind, then it induces a flop between $X_{q_1,O',0}$ and $X_{q_2,O',0}$. These flops are completely classified and we call them Mukai flops (cf. Definition 3.1). If it is of the second kind, the maps $X_{q_i,O',0} \to Z_0$ $(i = 1,2)$ are both divisorial birational maps. Define $\mathcal{S}^1(\mathfrak{l})$ to be the subset of $\mathcal{S}(\mathfrak{l})$ consisting of the parabolic subalgebras q' obtained from q by a finite succession of the twists of the first kind. Note that the restriction map $H^2(X_{q',O'}, \mathbb{R}) \to H^2(X_{q',O',0}, \mathbb{R})$ is an isomorphism and $\overline{\mathrm{Amp}}(\mu_{q'})$ is mapped onto $\overline{\mathrm{Amp}}(\mu_{q',0})$. We show:

Theorem 3.17. *There is a one-to-one correspondence between the set of $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$ and $\mathcal{S}^1(\mathfrak{l})$. In other words, every $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ is obtained as $\mu_{q',0} : X_{q,O',0} \to \bar{O}$ for $q' \in \mathcal{S}^1(\mathfrak{l})$. Two different $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$ are connected by a sequence of Mukai flops. Moreover*

$$\overline{\mathrm{Mov}}(\mu_{q,0}) = \cup_{q' \in \mathcal{S}^1(\mathfrak{l})} \overline{\mathrm{Amp}}(\mu_{q',0}),$$

where $\overline{\mathrm{Mov}}(\mu_{q,0})$ is the movable cone for $\mu_{q,0}$ (cf. 3.5).

A direct approach to Theorem 3.17 usually needs the classification of the generalized Springer maps which are isomorphisms in codimension one. But our approach using Theorem 3.14 does not need this and Mukai flops appear in a very natural way.

Let W be the Weyl group of $\mathfrak{g}$ and let $N_W(L)$ be the subgroup of W which normalizes L. Then the quotient group

$$W' := N_W(L)/W(L)$$

naturally acts on $M(L)_{\mathbf{R}}$. The interior $\mathrm{Mov}(\mu_{\mathfrak{q},0})$ of the movable cone can be characterized as a fundamental domain for this action (Theorem 3.18). The group W' was extensively studied in [11]. As explained above, the deformation $X_{\mathfrak{q},O'} \to \mathfrak{k}$ of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$ played an important role to study the birational geometry for $\tilde{O}$. But this is not merely a flat deformation of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. In fact, $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$ admits a symplectic 2-form on its regular locus. This symplectic 2-form induces a Poisson structure of the regular part; moreover, it uniquely extends to a Poisson structure of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. One can introduce the notion of a Poisson deformation (cf. §4), and $X_{\mathfrak{q},O'} \to \mathfrak{k}$ turns out to be a Poisson deformation of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. On the other hand, since $\tilde{O}$ has symplectic singularities, $\tilde{O}$ also admits a natural Poisson structure. One can construct a flat deformation of $\tilde{O}$ as follows. Let $G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}') \subset \mathfrak{g}$ denote the G-orbit of $\mathfrak{r}(\mathfrak{q}) + \bar{O}'$ by the adjoint G-action. By using the adjoint quotient map $\mathfrak{g} \to \mathfrak{h}/W$, we get a map $\chi : G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}') \to \mathfrak{h}/W$. The image of χ is not necessarily normal, but its normalization coincides with $\mathfrak{k}(\mathfrak{q})/W'$. Let $G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}')^n$ be the normalization of $G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}')$. Then χ induces a map

$$G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}')^n \to \mathfrak{k}(\mathfrak{q})/W'.$$

One can check that this is a flat map and its central fiber is $\tilde{O}$. Moreover, this is a Poisson deformation of $\tilde{O}$. The two Poisson deformations are combined together by the Brieskorn-Slodowy diagram

$$\begin{array}{ccc} X_{\mathfrak{q},O'} & \longrightarrow & G \cdot (\mathfrak{r}(\mathfrak{q}) + \bar{O}')^n \\ \downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{q}) & \longrightarrow & \mathfrak{k}(\mathfrak{q})/W' \end{array} \tag{1.1}$$

Theorem 4.5 claims that this gives formally universal Poisson deformations of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$ and $\tilde{O}$.

Finally we shall explain the contents of this paper. The first part of §2 is an introduction to nilpotent orbits and related resolutions. Many concrete examples are described in terms of flags; I believe that they would motivate the following abstract arguments. In the final part of §2, we give a rough sketch of the proof of Theorem 2.6 in the classical cases. The readers can find a proof in [8] when $\mathfrak{g}$ is exceptional. The idea of most arguments in §3 comes from [19]. But all statements are generalized so that one can apply them to generalized Springer maps. §4 is concerned with a Poisson deformation. We quickly review the notions of Poisson structures and Poisson deformations. After that, we will give a rough sketch of Theorem 4.5 mentioned above. The results of §4 have been already treated in [22] when $\tilde{O}$ has a crepant resolution.

Notations. Let G be an algebraic group over **C** and P a closed subgroup of G. If V is a variety with a P-action, then we denote by $G \times^P V$ the associated fiber bundle over G/P with a typical fiber V. More exactly, $G \times^P V$ is defined as the quotient of $G \times P$ by an equivalence relation $\sim$, where $(g, x) \sim (g', x')$ if there is an element $p \in P$ such that $g' = gp$ and $x' = p^{-1} \cdot x$.

2. Nilpotent orbits and symplectic singularities

2.1. Kostant-Kirillov form

Let G be a semi-simple algebraic group over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. An orbit O of the adjoint action $Ad : G \to Aut(\mathfrak{g})$ is called an adjoint orbit. Moreover, if O consists of nilpotent elements (resp. semi-simple elements), then O is called a *nilpotent orbit* (resp. *semi-simple orbit*). The tangent space $T_\alpha O$ of an adjoint orbit O at α is identified with

$$[\alpha, \mathfrak{g}] := \{[\alpha, x]; x \in \mathfrak{g}\}.$$

Since $\mathfrak{g}$ is semi-simple, the Killing form

$$k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

is a non-degenerate symmetric form. We define a skew-symmetric form

$$\omega_\alpha : T_\alpha O \times T_\alpha O \to \mathbb{C}$$

by

$$\omega_\alpha([\alpha, x], [\alpha, y]) := k(\alpha, [x, y]).$$

This is well-defined and non-degenerate because if $[\alpha, x] = 0$, then $k(\alpha, [x, y]) = k([\alpha, x], y) = 0$. If α runs through all elements of O, the 2-form $\omega := \{\omega_\alpha\}$ is a *d-closed* form on O. In particular, O is a smooth algebraic variety of even dimension. The symplectic form ω is called the Kostant-Kirillov 2-form. A semi-simple orbit is a closed subvariety of $\mathfrak{g}$. But, a nilpotent orbit O is not closed in $\mathfrak{g}$ except when $O = \{0\}$. If we take the closure $\bar{O}$ of O, it is an affine variety with singularities. Note that $\bar{O}$ is not necessarily normal. We denote by $\tilde{O}$ its normalization.

2.2. Nilpotent orbits in a classical Lie algebra

Let $sl(n)$ be the Lie algebra consisting of $n \times n$ matrices A with $tr(A) = 0$. Define

$$so(n) := \{A \in sl(n); A^t J + JA = 0\},$$