Edwin Hewitt
Kenneth A. Ross

Volume 115

Grundlehren
der mathematischen
Wissenschaften

A Series of
Comprehensive Studies
in Mathematics

Abstract Harmonic Analysis I

Second Edition

抽象调和分析 第1卷 第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

Edwin Hewitt Kenneth A. Ross

Abstract Harmonic Analysis

Volume I

Structure of Topological Groups
Integration Theory Group Representations

Second Edition

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

图书在版编目（CIP）数据

抽象调和分析．第1卷：第2版＝Abstract harmonic analysis：2nd ed：英文/（美）休伊特（Hewitt，E.）著．—影印本．—北京：世界图书出版公司北京公司，2013.10

ISBN 978－7－5100－7033－4

Ⅰ.①抽… Ⅱ.①休… Ⅲ.①抽象调和分析—英文 Ⅳ.①O177.5

中国版本图书馆CIP数据核字（2013）第249371号

书　　名：Abstract Harmonic Analysis 1 2nd ed.
作　　者：Edwin Hewitt，Kenneth A. Ross
中 译 名：抽象调和分析 第1卷 第2版
责任编辑：高蓉　刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河市国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街137号 100010）
联系电话：010－64021602，010－64015659
电子信箱：kjb@wpcbj.com.cn

开　　本：24开
印　　张：22.5
版　　次：2014年03月
版权登记：图字：01－2013－2915

书　　号：978－7－5100－7033－4　　定　　价：89.00元

Grundlehren der mathematischen Wissenschaften 115

A Series of Comprehensive Studies in Mathematics

Grundlehren der mathematischen Wissenschaften

A Series of Comprehensive Studies in Mathematics

2. Knopp: Theorie und Anwendung der unendlichen Reihe
3. Hurwitz: Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen
4. Madelung: Die mathematischen Hilfsmittel des Physikers
10. Schouten: Ricci-Calculus
14. Klein: Elementarmathematik vom höheren Standpunkte aus. 1. Band: Arithmetik, Algebra, Analysis
15. Klein: Elementarmathematik vom höheren Standpunkte aus. 2. Band: Geometrie
16. Klein: Elementarmathematik vom höheren Standpunkte aus. 3. Band: Präzisions- und Approximationsmathematik
20. Pólya/Szegö: Aufgaben und Lehrsätze aus der Analysis II: Funktionentheorie, Nullstellen, Polynome, Determinauten, Zahlentheorie
22. Klein: Vorlesungen über höhere Geometrie
26. Klein: Vorlesungen über nicht-euklidische Geometrie
27. Hilbert/Ackermann: Grundzüge der theoretischen Logik
30. Lichtenstein: Grundlagen der Hydromechanik
31. Kellogg: Foundations of Potential Theory
32. Reidemeister: Vorlesungen über Grundlagen der Geometrie
38. Neumann: Mathematische Grundlagen der Quantenmechanik
40. Hilbert/Bernays: Grundlagen der Mathematik I
43. Neugebauer: Vorlesungen über Geschichte der antiken mathematischen Wissenschaften. Band 1: Vorgriechische Mathematik
50. Hilbert/Bernays: Grundlagen der Mathematik II
52. Magnus/Oberhettinger/Soni: Formulas and Theorems for the Special Functions of Mathematical Physics
57. Hamel: Theoretische Mechanik
58. Blaschke/Reichardt: Einführung in die Differentialgeometrie
59. Hasse: Vorlesungen über Zahlentheorie
60. Collatz: The Numerical Treatment of Differential Equations
61. Maak: Fastperiodische Funktionen
62. Sauer: Anfangswertprobleme bei partiellen Differentialgleichungen
64. Nevanlinna: Uniformisierung
66. Bieberbach: Theorie der gewöhnlichen Differentialgleichungen
68. Aumann: Reelle Funktionen
69. Schmidt: Mathematische Gesetze der Logik I
71. Meixner/Schäfke: Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme
73. Hermes: Einführung in die Verbandstheorie
74. Boerner: Darstellungen von Gruppen
75. Rado/Reichelderfer: Continuous Transformations in Analysis, with an Introduction to Algebraic Topology
76. Tricomi: Vorlesungen über Orthogonalreihen
77. *Behnke/Sommer: Theorie der analytischen Funktionen einer komplexen Veränderlichen*
78. Lorenzen: Einführung in die operative Logik und Mathematik
80. Pickert: Projektive Ebenen

Edwin Hewitt
Department of Mathematics GN-50
University of Washington
Seattle, WA 98195, USA

Kenneth A. Ross
Department of Mathematics
University of Oregon
Eugene, OR 97403, USA

Mathematics Subject Classification (1991): 43A70, 22B05, 22A10, 43A15

Library of Congress Cataloging in Publication Data. Hewitt, Edwin, 1920–. Abstract harmonic analysis. (Grundlehren der mathematischen Wissenschaften; 115). Bibliography: p. Includes indexes.
ISBN 0-387-94190-8
CONTENTS: v. 1. Structure of topological groups, integration theory, group representations. 1. Harmonic analysis. I. Ross, Kenneth A., joint author. II. Title. III. Series: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen; 115.
QA403.H4. 1979. 515′.2433. 79-13097

ISBN 0-387-94190-8/1994 $5.00 + 0.20

Reprint from English language edition:
Abstract Harmonic Analysis I 2nd ed.
by Edwin Hewitt, Kenneth A. Ross
Springer-Verlag is a part of Springer Science+Business Media

Preface to the Second Edition

It has not been possible to rewrite the entire book for this Second Edition. It would have been gratifying to resurvey the theory of topological groups in the light of progress made in the period 1962—1978, to amplify some sections and curtail others, and in general to profit from our experience since the book was published. Market conditions and other commitments incurred by the authors have dictated otherwise. We have nonetheless taken advantage of the kindness of Springer-Verlag to make a number of improvements in the text and of course to correct misprints and mathematical blunders.

We are in debt to the readers who have written to us or spoken with us about the text, and we have tried to follow their suggestions. We are happy here to record our gratitude to ROBERT B. BURCKEL, W. WISTAR COMFORT, ROBERT E. EDWARDS, ROBERT E. JAMISON, JORGE M. LÓPEZ, THEODORE W. PALMER, WILLARD A. PARKER, KARL R. STROMBERG, and FRED THOELE, as well as to a host of others who have kindly made suggestions to us.

Our thanks are due as well to Springer-Verlag for their support of our work.

Seattle, Washington — EDWIN HEWITT
Eugene, Oregon — KENNETH A. ROSS

January 1979

Preface to the First Edition

When we accepted the kind invitation of Prof. Dr. F. K. SCHMIDT to write a monograph on abstract harmonic analysis for the *Grundlehren der Mathematischen Wissenschaften* series, we intended to write all that we could find out about the subject in a text of about 600 printed pages. We intended that our book should be accessible to beginners, and we hoped to make it useful to specialists as well. These aims proved to be mutually inconsistent. Hence the present volume comprises only half of the projected work. It gives all of the structure of topological groups needed for harmonic analysis as it is known to us; it treats integration on locally compact groups in detail; it contains an introduction to the theory of group representations. In the second volume we will treat harmonic analysis on compact groups and locally compact Abelian groups, in considerable detail.

The book is based on courses given by E. HEWITT at the University of Washington and the University of Uppsala, although naturally the material of these courses has been enormously expanded to meet the needs of a formal monograph. Like the other treatments of harmonic analysis that have appeared since 1940, the book is a lineal descendant of A. WEIL's fundamental treatise (WEIL [4])[1]. The debt of all workers in the field to WEIL's work is well known and enormous. We have also borrowed freely from LOOMIS's treatment of the subject (LOOMIS [2]), from NAĬMARK [1], and most especially from PONTRYAGIN [7]. In our exposition of the structure of locally compact Abelian groups and of the PONTRYAGIN-VAN KAMPEN duality theorem, we have been strongly influenced by PONTRYAGIN's treatment. We hope to have justified the writing of yet another treatise on abstract harmonic analysis by taking up recent work, by writing out the details of every important construction and theorem, and by including a large number of concrete examples and facts not available in other textbooks.

The book is intended to be readable by students who have had basic graduate courses in real analysis, set-theoretic topology, and algebra as given in United States universities at the present day. That is, we suppose that the reader knows elementary set theory, set-theoretic topology, measure theory, and algebra. Our ground rule is [although this is

[1] Here and throughout the book, numbers in square brackets designate works in the bibliography found at the end of the volume. These are arranged and numbered by authors.

not hard and fast] that facts and concepts from KELLEY [2], HALMOS [2], and VAN DER WAERDEN [1] may be used without explanation or proof.

In our effort to make the book useful for specialists, we have included a large corpus of material which a beginning reader may find it wise to omit. The following suggestions are for the beginning student who wishes to get to the root of the matter as quickly as possible. First read §§1—7, Subheads (8.1)—(8.7), (9.1)—(9.14), and all of §10. The reader who is already familiar with or wishes to take on faith the theory of integration on locally compact Hausdorff spaces may skip §§11—14. Section 15 is absolutely essential and should be read with careful attention. Sections 16—18 may be omitted. Sections 19 through 24 are vital, and should be read carefully. Sections 25 and 26 are rather special, but are so interesting that we hope all readers will work through them.

Sections 4—11 and 15—26 contain subsections entitled Miscellaneous Theorems and Examples. Some of these are worked out in detail; for others, proofs are sketched or omitted entirely. We refer *occasionally* in the main text to a result drawn from the Miscellaneous Theorems and Examples. All such results are easy and are supplied with proofs. The reader is counselled at least to read the statements of the Miscellaneous Theorems and Examples, and to use them as exercises *ad libitum*.

Many sections also contain historical notes. We have tried to trace the history of the principal theorems and concepts, but we obviously have not produced a complete history, and it would be foolish to claim that we have produced one correct in every detail. Also, while some of the results we give are new so far as we know, *failure to cite a reference for a given theorem should not be construed as a claim of originality on our part.*

For the reader's convenience, we have assembled in three appendices certain ancillary material not easily accessible elsewhere, which is essential for one part or another of the theory. These appendices may be read as the reader encounters references to them in the main text.

We have been assisted by many colleagues. W. WISTAR COMFORT, JOHN M. ERDMAN, ALBERT J. FRODERBERG, L. FUCHS, JAMES MICHELOW, RICHARD S. PIERCE, KARL R. STROMBERG, ELMAR THOMA, VEERAVALLI S. VARADARAJAN, and HERBERT S. ZUCKERMAN have all read various parts of the manuscript, made valuable improvements, and saved us on occasion from grievous error. Improvements have also been made by NORMAN J. BLOCH, FRANCIS X. CONNOLLY, GERALD L. ITZKOWITZ, and RICHARD T. SHANNON. For the help so generously given by our friends we are sincerely grateful. Our thanks are due to Mlles. LYNNE HARPER and JEANNE SLOPER for preparation of the typescript. Mr. ALBERT J. FRODERBERG has most generously assisted us in the task of proofreading.

The reading and research by both of us on which the book is based have been generously supported by the National Science Foundation, U.S.A., and by a fellowship of the John Simon Guggenheim Memorial Foundation granted to E. HEWITT. We extend our sincere thanks to the authorities of these foundations, without whose aid our work could not have been done.

Finally, we record our gratitude to Prof. Dr. F. K. SCHMIDT for his original suggestion that the book be written, and to the publishers for their dispatch, skill, and attention to our every wish in producing the book.

Seattle, Washington, Rochester, New York

EDWIN HEWITT
KENNETH A. ROSS

October 1962

Table of Contents

Chapter One

Preliminaries

Abstract harmonic analysis has evolved over the past few decades on the basis of several theories. First we have the classical theory of Fourier series and integrals, set forth in many treatises, such as ZYGMUND [1] and BOCHNER [1]. Second, we have the algebraic theory of groups and their representations, which is also described in many standard texts (*e.g.* VAN DER WAERDEN [1]). Third, we have the theory of topological spaces, by now a fundamental tool of analysis, and also the subject of standard texts (*e.g.* KELLEY [2]). The latter two subjects were combined to form the notion of a *topological group*. This is an entity which is both a group and a topological space and in which the group operations and the topology are appropriately connected. The structure of topological groups was extensively studied in the years 1925—1940; and the subject is far from dead even today.

Using a fundamental construction published in 1933 by A. HAAR [3], A. WEIL [4] in 1940 showed that Fourier series and integrals are but special cases of a construct which can be produced on a very wide class of topological groups. Furthermore, several classical theorems about Fourier series and integrals [PARSEVAL's equality, PLANCHEREL's theorem, the HERGLOTZ-BOCHNER theorem, and the HAUSDORFF-YOUNG inequality, to name a few] could be meaningfully stated, and proved, in this general situation. The end of the development initiated by WEIL is nowhere in sight. Problems in one branch or another of harmonic analysis are occupying the attention of many mathematicians at the present day, and we cannot predict at all where future developments will lead.

At any rate, the foundations of abstract harmonic analysis now seem to be clear. The present volume is devoted to a study of these foundations. Our first task is to learn thoroughly what a topological group is. We naturally need an array of facts about groups and topological spaces *per se*, which are set down in the present chapter.

§ 1. Notation and terminology

In this section, we explain some of the notation and terminology used throughout the text. Standard concepts and notation are used without explanation.

The symbols $\subset$ and $\supset$ mean ordinary inclusion between sets; they do not exclude the possibility of equality. The void set is denoted by $\varnothing$. Frequently the sets we deal with are subsets of some universal set, say E. In this case we denote by A' the *complement* of the set A:

$$A' = \{x : x \in E \text{ and } x \notin A\}.$$

In every case it will be clear what the set E is. For sets A and B, the *symmetric difference* $A \triangle B$ is defined to be the set $(A \cap B') \cup (A' \cap B)$.

A family $\mathscr{A}$ of sets has the *finite intersection property* if $\bigcap\{A : A \in \mathscr{F}\} \neq \varnothing$ for all finite subfamilies $\mathscr{F}$ of $\mathscr{A}$. In particular, no set in $\mathscr{A}$ is void if $\mathscr{A}$ has the finite intersection property. A collection $\{A_\iota\}_{\iota\in I}$ of sets is said to *partition* a set X if $\bigcup_{\iota\in I} A_\iota = X$, each A_ι is nonvoid, and the sets A_ι are pairwise disjoint. A family $\{A_\iota\}_{\iota\in I}$ of sets is a *covering* or *cover* of X if $\bigcup_{\iota\in I} A_\iota \supset X$. Given a set X, $\mathscr{P}(X)$ denotes the family of all subsets of X.

Knowledge of elementary cardinal and ordinal arithmetic is assumed. The family of *countable* sets includes finite sets and the void set. Infinite countable sets have cardinal number $\aleph_0$ and the real line has cardinal number $2^{\aleph_0}$. We will write $\mathfrak{c}$ for $2^{\aleph_0}$. The cardinal number of an arbitrary set A is denoted by $\overline{\overline{A}}$.

A subset Y of a partially ordered set X is said to be *cofinal* in X if for every $x \in X$, there is a $y \in Y$ such that $x \leqq y$.

The terms "mapping", "transformation", and "correspondence" are synonymous with "function". The term "operator" has a special meaning, which is explained in (B.2). A function f will often be defined by an expression

$$x \to f(x)$$

where x denotes a generic element of the domain of the function and $f(x)$ denotes its image under f. For a function f and a subset A of its domain, $f|A$ denotes the *restriction* of f to A.

If f is a function on X into Y and g is a function on Y into Z, then the *composition* of g by f is the function $g \circ f$ on X into Z defined by $g \circ f(x) = g\,(f(x))$ for $x \in X$.

Let X be a set and A any subset of X. The symbol ξ_A will denote the function defined on X such that

$$\xi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in A'. \end{cases}$$

The function ξ_A is called the *characteristic function* of A.

Let $\{X_\iota : \iota \in I\}$ be a nonvoid family of sets. We define $\mathsf{P}_{\iota\in I} X_\iota$ to be the set of all functions x from I into $\bigcup_{\iota\in I} X_\iota$ such that $x(\iota) \in X_\iota$ for all $\iota \in I$. This set is called the *Cartesian product* of the sets X_ι. We will almost invariably write the elements of $\underset{\iota\in I}{\mathsf{P}} X_\iota$ as (x_ι) where $x_\iota = x(\iota)$. Thus (x_ι) is an element of $\underset{\iota\in I}{\mathsf{P}} X_\iota$ and for each ι, x_ι is an element of X_ι. If the cardinal number of I is $\mathfrak{m}$ and if all of the sets X_ι are a fixed set X, we will frequently write $X^{\mathfrak{m}}$ for $\underset{\iota\in I}{\mathsf{P}} X_\iota$. If I is finite, say $I=\{1, 2, \ldots, m\}$, we sometimes write $\underset{\iota\in I}{\mathsf{P}} X_\iota$ as $\overset{m}{\underset{k=1}{\mathsf{P}}} X_k$ or $X_1 \times X_2 \times \cdots \times X_m$.

We reserve the symbols R for the set of all real numbers, K for the set of all complex numbers, Q for the set of all rational real numbers, and Z for the set of all integers. The symbol exp stands for the exponential function defined on K. The symbol T represents the set $\{\exp(2\pi i x): 0 \leqq x < 1\}$; it is a subset of K. The set of all numbers $\exp(2\pi i k/p^n)$, where p is a fixed prime, k runs through all integers, and n through all nonnegative integers, will be denoted by $Z(p^\infty)$. The function *signum* or sgn on K is defined by $\operatorname{sgn} z = \frac{z}{|z|}$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$.

For real numbers a and b, where $a \leqq b$, we define $[a, b] = \{x \in R : a \leqq x \leqq b\}$, $]a, b[= \{x \in R : a < x < b\}$, $]a, b] = \{x \in R : a < x \leqq b\}$, and $[a, b[= \{x \in R : a \leqq x < b\}$. We also define $]a, \infty[= \{x \in R : a < x\}$, with analogous definitions for $[a, \infty[$, $]-\infty, a[$, $]-\infty, a]$, and $]-\infty, \infty[$.

The sets K^n and R^n are *complex n-dimensional space* and *real n-dimensional space*, respectively ($n = 2, 3, 4, \ldots$). For $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ in K^n, we define the *inner product* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ as $\sum_{k=1}^{n} a_k \overline{b_k}$ and the *norm* $\|\boldsymbol{a}\|$ as $\langle \boldsymbol{a}, \boldsymbol{a} \rangle^{\frac{1}{2}}$. Let S_{n-1} denote the set of all $\boldsymbol{a} \in R^n$ such that $\|\boldsymbol{a}\| = 1$.

For a nonvoid set X and a positive real number p, we define $l_p(X)$ as the set of all complex-valued functions a on X such that $\sum_{x\in X} |a(x)|^p < \infty$. Obviously a function in $l_p(X)$ vanishes outside of a countable subset of X. The *norm* of $a \in l_p(X)$ is the number

$$\Big[\sum_{x\in X} |a(x)|^p\Big]^{1/p}.$$

See § 12 for a complete discussion.

For an arbitrary nonvoid set X, let δ_{xy} denote the function on $X \times X$ such that $\delta_{xx} = 1$ for all $x \in X$ and $\delta_{xy} = 0$ if $x \neq y$ and $x, y \in X$. This function is called *Kronecker's delta function*.

The end of each proof is indicated by the symbol □.

§ 2. Group theory

In this section, we establish the terminology and notation concerning groups that will be used throughout the book. We also prove a few theorems about groups, although most of these are quite standard[1]. We

[1] The only nonstandard theorem in § 2 is (2.9).

do this to clarify analogous theorems appearing in subsequent sections, where topological considerations also play a rôle. For a very detailed account of the theory of groups, the reader is referred to KUROŠ [1]; for a shorter and more elementary version, see for example VAN DER WAERDEN [1].

For the detailed analysis given in §§ 9, 23—26, we require many refined theorems about Abelian groups. These are given in Appendix A, with complete proofs.

In dealing with abstract groups, we will adhere to multiplicative notation for the group operation, except in part of Appendix A, where it is convenient to write the group operation as addition. In dealing with R, Z, R^n, etc., which are obviously groups under addition, we will write the group operation as addition. In discussions involving a multiplicative group G, the symbol e will be reserved for the identity element of G.

We are principally concerned with groups in this book. For some purposes, however, it is useful to consider semigroups. A *semigroup* is a nonvoid set G and a mapping $(x,y)\to xy$ of $G\times G$ into G such that $x(yz)=(xy)z$ for all x, y, z in G. That is, a semigroup is any nonvoid set with an associative multiplication. We do *not* assume the existence of an identity or the validity of any cancellation law. Note too that semigroups may contain subgroups and that groups may contain subsemigroups that are not groups.

Let G be a group. For a fixed $a\in G$, the mappings $x\to ax$ and $x\to xa$ of G onto itself are called *left* and *right translation* by the element a, respectively. The mapping $x\to x^{-1}$ of G onto itself is called *inversion*.

Mappings $x\to axa^{-1}$ of G onto itself are called *inner automorphisms* of G. We will frequently write inner automorphisms as ϱ_a: $\varrho_a(x)=axa^{-1}$. The set of automorphisms of a group G is itself a group under the operation of composition. The set of all inner automorphisms of G forms a subgroup of the group of all automorphisms of G; and the mapping $a\to\varrho_a$ is a homomorphism of G into its automorphism group.

Let A and B be subsets of a group G. The symbol AB denotes the set $\{ab: a\in A,\ b\in B\}$, and A^{-1} denotes $\{a^{-1}: a\in A\}$. We write aB for $\{a\}B$ and Ba for $B\{a\}$. We write AA as A^2, AAA as A^3, etc. Also we write $A^{-1}A^{-1}$ as A^{-2}, etc. A subgroup H of G such that $H\neq G$ and $H\neq\{e\}$ is called a *proper* subgroup of G.

For a subgroup H of a group G, the symbol G/H will *always* denote the space of *left* cosets of H in G. Thus points of G/H are sets xH, and $G/H=\{xH: x\in G\}$. Let H be a normal subgroup of the group G. The mapping $x\to xH$ of G onto G/H is called the *natural mapping of G onto G/H*. It is obviously a homomorphism. The group G/H is called the *quotient group* of G by H. For an element a of G, let ${}_a\psi$ be the mapping of G/H onto itself defined by ${}_a\psi(xH) = (ax)H$, for all $xH \in G/H$. It is clear that ${}_a\psi$ is well defined, that ${}_a\psi$ is a one-to-one mapping of G/H onto itself, and that the set $\{{}_a\psi : a \in G\}$ forms a group under composition that is a homomorphic image of G under the mapping $a \to {}_a\psi$. Also, given cosets xH and yH, the mapping ${}_{yx^{-1}}\psi$ obviously carries xH onto yH.

Two elements a and b in G are said to be *conjugate* if some inner automorphism maps a onto b, *i.e.* if $a = xbx^{-1}$ for some $x \in G$. There exists a partition $\{A_\iota\}_{\iota \in I}$ of G such that two elements are conjugate if and only if they belong to the same A_ι; the sets A_ι are called the *conjugacy classes* of G.

As already noted, $R, Z, Q, R^n, K^n, \ldots$ are Abelian groups under addition. Observe also that T and $Z(p^\infty)$ are Abelian groups under multiplication. The groups R, T, Q, and $Z(p^\infty)$ are of special importance in the structure theory developed in §§ 9, 24—26. The group T in addition is of vital importance in harmonic analysis: this is explained in §§ 23 and 24.

The symbol $Z(m)$ denotes the finite cyclic group of m elements $(m = 2, 3, \ldots)$: we will frequently represent this group as the set of integers $\{0, 1, 2, \ldots, m-1\}$, with addition modulo m.

(2.1) First isomorphism theorem. *Let G be a group, H a normal subgroup of G, and A an arbitrary subgroup of G. Then $AH = HA$ is a subgroup of G, H is a normal subgroup of AH, and $H \cap A$ is a normal subgroup of A. The groups $(AH)/H$ and $A/(H \cap A)$ are isomorphic. In fact, the mapping τ defined by $aH \to (aH) \cap A = a(H \cap A)$, $a \in A$, is an isomorphism of $(AH)/H$ onto $A/(H \cap A)$.*

Proof. It is obvious that $AH = HA$, that HA is a subgroup of G, that H is a normal subgroup of HA, and that $H \cap A$ is a normal subgroup of A. Consider the mapping of A onto $(AH)/H$ defined by $a \to aH$. It is easy to see that this mapping is a homomorphism and that its kernel is $H \cap A$. By the fundamental homomorphism theorem for groups, AH/H and $A/(H \cap A)$ are isomorphic and the isomorphism can be given by $a(H \cap A) \to aH$; let τ be the inverse of this mapping. □

(2.2) Second isomorphism theorem. *Let G and $\widetilde{G}$ be groups with identity elements e and $\tilde{e}$, respectively, and let φ be a homomorphism of G onto $\widetilde{G}$. Let $\widetilde{H}$ be any normal subgroup of $\widetilde{G}$, $H = \varphi^{-1}(\widetilde{H})$, and $N = \varphi^{-1}(\tilde{e})$. Then the groups G/H, $\widetilde{G}/\widetilde{H}$, and $(G/N)/(H/N)$ are isomorphic.*

Proof. The mapping $x \to \varphi(x)\widetilde{H}$ is a homomorphism of G onto $\widetilde{G}/\widetilde{H}$ with kernel H. Hence G/H is isomorphic with $\widetilde{G}/\widetilde{H}$. Since $\widetilde{G}$ is isomorphic with G/N and $\widetilde{H}$ is isomorphic with H/N, $\widetilde{G}/\widetilde{H}$ is isomorphic with $(G/N)/(H/N)$. □

(2.3) Direct products. Let $\{G_\iota : \iota \in I\}$ be a nonvoid family of groups and let $\underset{\iota\in I}{\mathsf{P}} G_\iota$ be the Cartesian product of the sets G_ι. For (x_ι) and (y_ι) in $\underset{\iota\in I}{\mathsf{P}} G_\iota$, let $(x_\iota)(y_\iota)$ be the element $(x_\iota y_\iota)$ in $\underset{\iota\in I}{\mathsf{P}} G_\iota$. Under this multiplication, $\underset{\iota\in I}{\mathsf{P}} G_\iota$ is a group; it is called the *direct product* of the groups G_ι. The groups G_ι are called *factors*. The identity of $\underset{\iota\in I}{\mathsf{P}} G_\iota$ is (e_ι), where each e_ι is the identity of G_ι. Let $\underset{\iota\in I}{\mathsf{P}^*} G_\iota$ be the set of all $(x_\iota) \in \underset{\iota\in I}{\mathsf{P}} G_\iota$ such that $x_\iota = e_\iota$ for all but a finite set of indices [this set varying with (x_ι)]. Then $\underset{\iota\in I}{\mathsf{P}^*} G_\iota$ is a subgroup of $\underset{\iota\in I}{\mathsf{P}} G_\iota$; it is called the *weak direct product* of the groups G_ι. If the cardinal number of I is $\mathfrak{m}$ and if each $G_\iota = G$, then we will write $G^{\mathfrak{m}}$ for $\underset{\iota\in I}{\mathsf{P}} G_\iota$ and $G^{\mathfrak{m}*}$ for $\underset{\iota\in I}{\mathsf{P}^*} G_\iota$.

(2.4) Theorem. *Let G be a group with identity e, and let $N_1, N_2, \ldots, N_m$ be a collection of normal subgroups of G satisfying:*

(i) $N_1 N_2 \cdots N_m = G$;

(ii) $(N_1 N_2 \cdots N_k) \cap N_{k+1} = \{e\}$ *for* $k = 1, 2, \ldots, m-1$.

Then G is isomorphic with the direct product $\overset{m}{\underset{k=1}{\mathsf{P}}} N_k$.

Proof. Every element x of G can be written as $x_1 x_2 \cdots x_m$ where each x_k belongs to N_k. By virtue of (ii), this representation is unique. Hence for $x \in G$, we define $\tau(x) = (x_k) \in \overset{m}{\underset{k=1}{\mathsf{P}}} N_k$ where $x_1 x_2 \cdots x_m = x$. It is then simple to show that τ is an isomorphism of G onto $\overset{m}{\underset{k=1}{\mathsf{P}}} N_k$. □

(2.5) Theorem. *Let G be a group with identity e, and let $\{N_\iota : \iota \in I\}$ be a nonvoid family of normal subgroups of G. For each $\iota \in I$, let M_ι be the smallest subgroup of G containing all N_λ for $\lambda \neq \iota$. Suppose that:*

(i) *the smallest subgroup of G containing all N_ι is G itself;*

(ii) *for every $\iota \in I$, the relation $M_\iota \cap N_\iota = \{e\}$ obtains.*

Then G is isomorphic with the weak direct product $\underset{\iota\in I}{\mathsf{P}^*} N_\iota$.

Proof. For $x_j \in N_{\iota_j}$ $(j = 1, 2)$, we have $x_1 x_2 x_1^{-1} x_2^{-1} = (x_1 x_2 x_1^{-1}) x_2^{-1} \in N_{\iota_2}$, and similarly $x_1 x_2 x_1^{-1} x_2^{-1} \in N_{\iota_1}$. Hence (ii) shows that $x_1 x_2 = x_2 x_1$. By (i) and the foregoing, every element of G can be written in the form $x_1 x_2 \cdots x_k$ where $x_j \in N_{\iota_j}$ $(j = 1, 2, \ldots, k)$. This representation is unique. For, if $x_1 x_2 \cdots x_k = y_1 y_2 \cdots y_k$ $(x_j, y_j \in N_{\iota_j})$, then $y_1^{-1} x_1 \cdots y_{k-1}^{-1} x_{k-1} = y_k x_k^{-1}$, and (ii) implies that $y_k = x_k$. It is now obvious that the mapping $x_1 x_2 \cdots x_k \to (y_\iota)$ [where $y_{\iota_j} = x_j$ for $j = 1, 2, \ldots, k$ and $y_\iota = e$ otherwise] is an isomorphism of G onto $\underset{\iota\in I}{\mathsf{P}^*} N_\iota$. □

(2.6) Semidirect products. Let L be a group, and suppose that L contains a normal subgroup G and a subgroup H such that $GH = L$ and $G \cap H = \{e\}$. That is, suppose that one can select exactly one element h from each coset of G so that $\{h\}$ forms a subgroup, H. If H is also normal,