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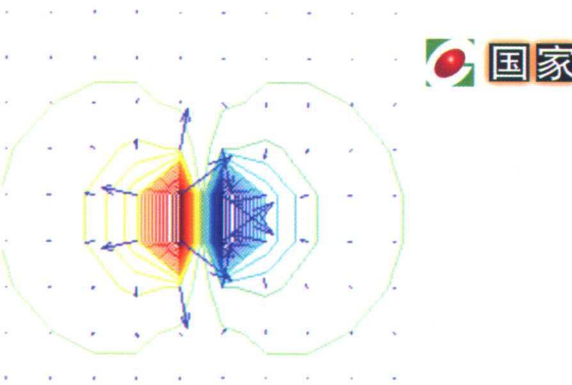

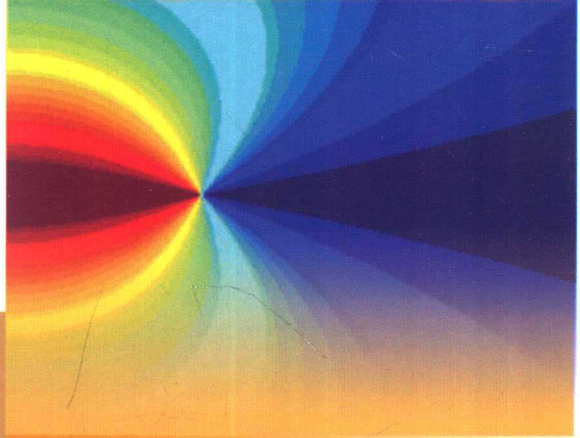

电磁场与电磁波习题解答

◎ 李一玫　编著

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内 容 简 介

本书为《电磁场与电磁波》(邵小桃、李一玫、王国栋编著)配套编写,可为使用该教材的教师和学生提供学习辅导。全书共七章,每章包括主要内容和习题及解答两部分。其中,主要内容概括了本章的内容提要和公式,习题及解答则对教材中的全部习题做了详尽分析和解答。依照作者多年的教学经验,在书中力求题目分析详尽,解答过程有序,问题总结及时,使之符合学生的学习特点。

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投诉电话：010－51686043，51686008；传真：010－62225406；E-mail：press@bjtu.edu.cn。

目　　录

第1章 矢量分析

1.1 主要内容

1. 标量场和矢量场的概念

一个函数能在空间某区域中各点表征一个物理存在称为一个场。标量场在区域中各点的物理特性用一个数来描述;矢量场则对区域中各点的物理特性同时用大小和方向来描述。

2. 矢量的标量积和矢量积

所有的矢量加法、乘法、微分运算都是对同一点进行的。

点积(标量积): $\boldsymbol{A}\cdot\boldsymbol{B}=AB\cos\theta$

直角坐标 $\boldsymbol{A}\cdot\boldsymbol{B}=A_xB_x+A_yB_y+A_zB_z$

圆柱坐标 $\boldsymbol{A}\cdot\boldsymbol{B}=A_\rho B_\rho+A_\varphi B_\varphi+A_zB_z$

球 坐 标 $\boldsymbol{A}\cdot\boldsymbol{B}=A_rB_r+A_\theta B_\theta+A_\varphi B_\varphi$

叉积(矢量积): $\boldsymbol{A}\times\boldsymbol{B}=|AB\sin\theta|\boldsymbol{a}_n$

直角坐标
$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

圆柱坐标
$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{a}_\rho & \boldsymbol{a}_\varphi & \boldsymbol{a}_z\\ A_\rho & A_\varphi & A_z\\ B_\rho & B_\varphi & B_z\end{vmatrix}$$

球 坐 标
$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{a}_r & \boldsymbol{a}_\theta & \boldsymbol{a}_\varphi\\ A_r & A_\theta & A_\varphi\\ B_r & B_\theta & B_\varphi\end{vmatrix}$$

3. 梯度、散度、旋度和拉普拉斯微分

标量场的梯度: $\nabla\Phi$

直角坐标
$$\nabla\Phi=\boldsymbol{a}_x\frac{\partial\Phi}{\partial x}+\boldsymbol{a}_y\frac{\partial\Phi}{\partial y}+\boldsymbol{a}_z\frac{\partial\Phi}{\partial z}$$

圆柱坐标
$$\nabla\Phi=\boldsymbol{a}_\rho\frac{\partial\Phi}{\partial\rho}+\boldsymbol{a}_\varphi\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}+\boldsymbol{a}_z\frac{\partial\Phi}{\partial z}$$

球 坐 标 $\nabla\Phi=\boldsymbol{a}_r\frac{\partial\Phi}{\partial r}+\boldsymbol{a}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta}+\boldsymbol{a}_\varphi\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}$

矢量场的散度：$\nabla\cdot\boldsymbol{A}$

直角坐标 $\nabla\cdot\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$

圆柱坐标 $\nabla\cdot\boldsymbol{A}=\frac{\partial A_\rho}{\partial\rho}+\frac{A_\rho}{\rho}+\frac{\partial A_\varphi}{\rho\,\partial\varphi}+\frac{\partial A_z}{\partial z}$

球 坐 标 $\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$

矢量场的旋度：$\nabla\times\boldsymbol{A}$

直角坐标 $$\nabla\times\boldsymbol{A}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

圆柱坐标 $$\nabla\times\boldsymbol{A}=\frac{1}{\rho}\begin{vmatrix}\boldsymbol{a}_\rho & \rho\,\boldsymbol{a}_\varphi & \boldsymbol{a}_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\varphi & A_z\end{vmatrix}$$

球坐标 $$\nabla\times\boldsymbol{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\boldsymbol{a}_r & r\,\boldsymbol{a}_\theta & r\sin\theta\,\boldsymbol{a}_\varphi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta\,A_\varphi\end{vmatrix}$$

标量场的拉普拉斯微分：$\nabla^2\Phi$

直角坐标 $\nabla^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}$

圆柱坐标 $\nabla^2\Phi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}$

球坐标 $\nabla^2\Phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$

4. 定理和矢量恒等式

散度定理： $\oint_S\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=\int_\tau\nabla\cdot\boldsymbol{A}\mathrm{d}\tau$

斯托克斯定理： $\oint_C\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}=\int_S\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$

格林第一恒等式： $\int_\tau\nabla\Phi\cdot\nabla\Psi\mathrm{d}\tau+\int_\tau\Phi\nabla^2\Psi\mathrm{d}\tau=\oint_S\Phi\nabla\Psi\cdot\mathrm{d}\boldsymbol{S}$

格林定理：$\int_\tau \Phi \nabla^2 \Psi \mathrm{d}\tau - \int_\tau \Psi \nabla^2 \Phi \mathrm{d}\tau = \oint_S (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot \mathrm{d}\boldsymbol{S}$

亥姆霍兹定理：$\boldsymbol{F}(\boldsymbol{r}) = -\nabla \Phi(\boldsymbol{r}) + \nabla \times \boldsymbol{A}(\boldsymbol{r})$

式中 $$\Phi(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\tau'} \frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\tau'$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\tau'} \frac{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\tau'$$

矢量斯托克斯定理：$\int_\tau (\nabla \times \boldsymbol{A})\mathrm{d}\tau = -\oint_S \boldsymbol{A} \times \mathrm{d}\boldsymbol{S}$

二阶微分：
$$\nabla \cdot \nabla \times \boldsymbol{A} = 0$$
$$\nabla \times \nabla \Phi = \boldsymbol{0}$$
$$\nabla \cdot \nabla \Phi = \nabla^2 \Phi$$
$$\nabla^2 \boldsymbol{A} = \nabla\nabla \cdot \boldsymbol{A} - \nabla \times \nabla \times \boldsymbol{A}$$

含标量乘积的微分：
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla \cdot (f\boldsymbol{A}) = f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$$
$$\nabla \times (f\boldsymbol{A}) = f\nabla \times \boldsymbol{A} + \nabla f \times \boldsymbol{A}$$

含矢量积的微分：$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\nabla \cdot \boldsymbol{B} - \boldsymbol{B}\nabla \cdot \boldsymbol{A} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$$

混合积：$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$

二重矢量积：$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$

1.2 习题及解答

1-1 已知矢量 $\boldsymbol{A} = \boldsymbol{a}_x 2 - \boldsymbol{a}_y 3 + \boldsymbol{a}_z 4$；$\boldsymbol{B} = \boldsymbol{a}_x 3 + \boldsymbol{a}_y 2 + \boldsymbol{a}_z$。求矢量 $\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{A}$ 的模、方向余弦及单位矢量。使用 MATLAB 检查答案。

【解】 $\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{A} = \boldsymbol{a}_x(3-2) + \boldsymbol{a}_y(2+3) + \boldsymbol{a}_z(1-4) = \boldsymbol{a}_x + \boldsymbol{a}_y 5 - \boldsymbol{a}_z 3$

$$C = \sqrt{1+5^2+(-3)^2} = \sqrt{35}$$

$$\cos\alpha = 1/\sqrt{35} \quad \cos\beta = 5/\sqrt{35} \quad \cos\gamma = -3/\sqrt{35}$$

$$\boldsymbol{a}_C = \frac{\boldsymbol{C}}{C} = \boldsymbol{a}_x \frac{1}{\sqrt{35}} + \boldsymbol{a}_y \frac{5}{\sqrt{35}} - \boldsymbol{a}_z \frac{3}{\sqrt{35}}$$

1-2 已知三个矢量分别为 $\boldsymbol{A} = \boldsymbol{a}_x + \boldsymbol{a}_y 2 - \boldsymbol{a}_z 3$；$\boldsymbol{B} = -\boldsymbol{a}_y 4 + \boldsymbol{a}_z$；$\boldsymbol{C} = \boldsymbol{a}_x 5 - \boldsymbol{a}_z 2$。试求：(1) $|\boldsymbol{A}|$，$|\boldsymbol{B}|$，$|\boldsymbol{C}|$；(2)单位矢量 $\boldsymbol{a}_A$，$\boldsymbol{a}_B$，$\boldsymbol{a}_C$；(3) $\boldsymbol{A} \cdot \boldsymbol{B}$，$\boldsymbol{A} \cdot \boldsymbol{C}$；(4) $\boldsymbol{A} \times \boldsymbol{B}$；(5) $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$ 及 $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$；(6) $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}$ 及 $(\boldsymbol{A} \times \boldsymbol{C}) \cdot \boldsymbol{B}$。使用 MATLAB 检查答案。

【解】 (1) $|\boldsymbol{A}| = \sqrt{1+2^2+(-3)^2} = \sqrt{14}$

$$|\boldsymbol{B}| = \sqrt{(-4)^2+1} = \sqrt{17}$$

$$|\boldsymbol{C}| = \sqrt{5^2+(-2)^2} = \sqrt{29}$$

(2) $\boldsymbol{a}_A = \frac{1}{\sqrt{14}}(\boldsymbol{a}_x + \boldsymbol{a}_y 2 - \boldsymbol{a}_z 3)$ $\boldsymbol{a}_B = \frac{1}{\sqrt{17}}(-\boldsymbol{a}_y 4 + \boldsymbol{a}_z)$ $\boldsymbol{a}_C = \frac{1}{\sqrt{29}}(\boldsymbol{a}_x 5 - \boldsymbol{a}_z 2)$

(3) $\boldsymbol{A}\cdot\boldsymbol{B} = -2\times4-3=-11$ $\boldsymbol{A}\cdot\boldsymbol{C} = 5-3\times(-2)=11$

(4) $\boldsymbol{A}\times\boldsymbol{B} = (\boldsymbol{a}_x+\boldsymbol{a}_y 2-\boldsymbol{a}_z 3)\times(-\boldsymbol{a}_y 4+\boldsymbol{a}_z) = -\boldsymbol{a}_x 10-\boldsymbol{a}_y-\boldsymbol{a}_z 4$

(5) $(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = (-\boldsymbol{a}_x 10-\boldsymbol{a}_y-\boldsymbol{a}_z 4)\times(\boldsymbol{a}_x 5-\boldsymbol{a}_z 2) = \boldsymbol{a}_x 2-\boldsymbol{a}_y 40+\boldsymbol{a}_z 5$

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{a}_x+\boldsymbol{a}_y 2-\boldsymbol{a}_z 3)\times(\boldsymbol{a}_x 8+\boldsymbol{a}_y 5+\boldsymbol{a}_z 20) = \boldsymbol{a}_x 55-\boldsymbol{a}_y 44-\boldsymbol{a}_z 11$

(6) $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = (-\boldsymbol{a}_x 10-\boldsymbol{a}_y-\boldsymbol{a}_z 4)\cdot(\boldsymbol{a}_x 5-\boldsymbol{a}_z 2) = -42$

$(\boldsymbol{A}\times\boldsymbol{C})\cdot\boldsymbol{B} = (-\boldsymbol{a}_x 4-\boldsymbol{a}_y 13-\boldsymbol{a}_z 10)\cdot(-\boldsymbol{a}_y 4+\boldsymbol{a}_z) = 42$

1-3 证明：$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$。

证明：利用恒等式 $\boldsymbol{X}\cdot(\boldsymbol{C}\times\boldsymbol{D}) = \boldsymbol{D}\cdot(\boldsymbol{X}\times\boldsymbol{C})$

和
$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C})$$

可得
$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) &= \boldsymbol{D}\cdot[(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}] = \boldsymbol{D}\cdot[\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C})]\\ &= (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})\end{aligned}$$

1-4 证明：$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{B}\times\boldsymbol{C})\times(\boldsymbol{C}\times\boldsymbol{A}) = [\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]^2$。

证明：利用恒等式 $\boldsymbol{X}\times(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}(\boldsymbol{X}\cdot\boldsymbol{A})-\boldsymbol{A}(\boldsymbol{X}\cdot\boldsymbol{C})$，令 $\boldsymbol{X}=\boldsymbol{B}\times\boldsymbol{C}$，有

$$\begin{aligned}(\boldsymbol{B}\times\boldsymbol{C})\times(\boldsymbol{C}\times\boldsymbol{A}) &= \boldsymbol{C}[(\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}]-\boldsymbol{A}[(\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{C}]\\ &= \boldsymbol{C}[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]-\boldsymbol{A}[\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{C})] = \boldsymbol{C}[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]\end{aligned}$$

于是

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{B}\times\boldsymbol{C})\times(\boldsymbol{C}\times\boldsymbol{A}) &= (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})] = [(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}][(\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]\\ &= [\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]^2\end{aligned}$$

1-5 已知空间三角形的顶点坐标为 $O(0,0,0)$，$P_1(1,4,3)$，$P_2(4,2,-4)$。试问：(1)该三角形是否直角三角形？(2)计算该三角形的面积。使用 MATLAB 检查答案。

【解】 (1) $\overrightarrow{OP_1} = \boldsymbol{a}_x+\boldsymbol{a}_y 4+\boldsymbol{a}_z 3$ $\overrightarrow{OP_2} = \boldsymbol{a}_x 4+\boldsymbol{a}_y 2-\boldsymbol{a}_z 4$ $\overrightarrow{P_1P_2} = \boldsymbol{a}_x 3-\boldsymbol{a}_y 2-\boldsymbol{a}_z 7$

$\overrightarrow{OP_1}\cdot\overrightarrow{OP_2} = 4+4\times2-3\times4=0$，即 $\overrightarrow{OP_1}\perp\overrightarrow{OP_2}$，因此该三角形是直角三角形。

(2) $S = \frac{1}{2}|\overrightarrow{OP_1}|\cdot|\overrightarrow{OP_2}| = \frac{1}{2}\sqrt{1+16+9}\sqrt{16+4+16} = 3\sqrt{26}$

1-6 已知矢量 $\boldsymbol{A} = \boldsymbol{a}_x 2+\boldsymbol{a}_y 3-\boldsymbol{a}_z 4$；$\boldsymbol{B} = -\boldsymbol{a}_x 6-\boldsymbol{a}_y 4+\boldsymbol{a}_z$；$\boldsymbol{C} = \boldsymbol{a}_x-\boldsymbol{a}_y+\boldsymbol{a}_z$。试求：

(1) $\boldsymbol{A}$、$\boldsymbol{B}$ 之间的夹角；(2) $\boldsymbol{A}\times\boldsymbol{B}$ 在 $\boldsymbol{C}$ 上的投影。使用 MATLAB 检查答案。

【解】 (1) $\cos\theta = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{-2\times6-3\times4-4\times1}{\sqrt{2^2+3^2+(-4)^2}\sqrt{(-6)^2+(-4)^2+1}} = -\frac{28}{\sqrt{29}\sqrt{53}} \approx -0.7142$

$$\theta = \arccos(-0.7142) \approx 180^\circ-44.42^\circ = 135.58^\circ;$$

(2) $\boldsymbol{A}\times\boldsymbol{B} = (\boldsymbol{a}_x 2+\boldsymbol{a}_y 3-\boldsymbol{a}_z 4)\times(-\boldsymbol{a}_x 6-\boldsymbol{a}_y 4+\boldsymbol{a}_z) = -\boldsymbol{a}_x 13+\boldsymbol{a}_y 22+\boldsymbol{a}_z 10$

$$\boldsymbol{a}_C = \frac{1}{\sqrt{3}}(\boldsymbol{a}_x-\boldsymbol{a}_y+\boldsymbol{a}_z)$$

$\boldsymbol{A}\times\boldsymbol{B}$ 在 $\boldsymbol{C}$ 方向上的投影为：

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{a}_C=(-\boldsymbol{a}_x13+\boldsymbol{a}_y22+\boldsymbol{a}_z10)\times\frac{1}{\sqrt{3}}(\boldsymbol{a}_x-\boldsymbol{a}_y+\boldsymbol{a}_z)=-\frac{25}{\sqrt{3}}$$

1-7 求以矢量 $\boldsymbol{A}=-\boldsymbol{a}_x2-\boldsymbol{a}_y3+\boldsymbol{a}_z$，$\boldsymbol{B}=\boldsymbol{a}_x2-\boldsymbol{a}_y5+\boldsymbol{a}_z3$ 和 $\boldsymbol{C}=\boldsymbol{a}_x4-\boldsymbol{a}_y2+\boldsymbol{a}_z6$ 为邻边构成的平行六面体的体积。使用 MATLAB 检查答案。

【解】
$$V=(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\begin{vmatrix}4&-2&6\\-2&-3&1\\2&-5&3\end{vmatrix}=64$$

1-8 计算在圆柱坐标系中 $P(5,\pi/6,5)$ 和 $Q(2,\pi/3,4)$ 两点之间的距离。

【解】 把圆柱坐标化为直角坐标，然后求距离：

$$\overrightarrow{OP}=\boldsymbol{a}_x5\cos\frac{\pi}{6}+\boldsymbol{a}_y5\sin\frac{\pi}{6}+\boldsymbol{a}_z5=\boldsymbol{a}_x\frac{5\sqrt{3}}{2}+\boldsymbol{a}_y\frac{5}{2}+\boldsymbol{a}_z5$$

$$\overrightarrow{OQ}=\boldsymbol{a}_x2\cos\frac{\pi}{3}+\boldsymbol{a}_y2\sin\frac{\pi}{3}+\boldsymbol{a}_z4=\boldsymbol{a}_x+\boldsymbol{a}_y\sqrt{3}+\boldsymbol{a}_z4$$

$$R=|\overrightarrow{OP}-\overrightarrow{OQ}|=\sqrt{\left(\frac{5\sqrt{3}}{2}-1\right)^2+\left(\frac{5}{2}-\sqrt{3}\right)^2+(5-4)^2}=\sqrt{30-10\sqrt{3}}\approx3.56$$

1-9 求球坐标中 $P(10,\pi/4,\pi/3)$ 和 $Q(2,\pi/2,\pi)$ 两点之间的距离，并求从 P 到 Q 的距离矢量。

【解】 把球坐标化为直角坐标，然后求距离：

$$\overrightarrow{OP}=\boldsymbol{a}_x10\sin\frac{\pi}{4}\cos\frac{\pi}{3}+\boldsymbol{a}_y10\sin\frac{\pi}{4}\sin\frac{\pi}{3}+\boldsymbol{a}_z10\cos\frac{\pi}{4}$$

$$=\boldsymbol{a}_x\frac{5\sqrt{2}}{2}+\boldsymbol{a}_y\frac{5\sqrt{6}}{2}+\boldsymbol{a}_z5\sqrt{2}$$

$$\overrightarrow{OQ}=\boldsymbol{a}_x2\sin\frac{\pi}{2}\cos\pi+\boldsymbol{a}_y2\sin\frac{\pi}{2}\sin\pi+\boldsymbol{a}_z2\cos\frac{\pi}{2}=-\boldsymbol{a}_x2$$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\boldsymbol{a}_x\left(-2-\frac{5\sqrt{2}}{2}\right)-\boldsymbol{a}_y\frac{5\sqrt{6}}{2}-\boldsymbol{a}_z5\sqrt{2}$$

$$R=|\overrightarrow{PQ}|=\sqrt{\left(-\frac{5\sqrt{2}}{2}-2\right)^2+\left(-\frac{5\sqrt{6}}{2}\right)^2+(-5\sqrt{2})^2}=\sqrt{104+10\sqrt{2}}\approx10.87$$

1-10 求点 $(6,-4,4)$ 至连接点 $(2,1,2)$ 与点 $(3,-1,4)$ 之直线的最短距离。

【解】 三个点依次连接构成一个三角形，该三角形的面积可由任意两条边的矢量积模值的一半求出，也可由其中的一条边长作为三角形的底，对应的高即是点到直线的最短距离，从而利用三角形的面积公式计算。

设 $A(6,-4,4)$，$B(2,1,2)$，$C(3,-1,4)$，则：

$$\overrightarrow{AB}=-\boldsymbol{a}_x4+\boldsymbol{a}_y5-\boldsymbol{a}_z2\quad\overrightarrow{AC}=-\boldsymbol{a}_x3+\boldsymbol{a}_y3\quad\overrightarrow{BC}=\boldsymbol{a}_x-\boldsymbol{a}_y2+\boldsymbol{a}_z2$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(-\boldsymbol{a}_x4+\boldsymbol{a}_y5-\boldsymbol{a}_z2)\times(-\boldsymbol{a}_x3+\boldsymbol{a}_y3)=\boldsymbol{a}_x6+\boldsymbol{a}_y6+\boldsymbol{a}_z3$$

由 $S_\Delta=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|\overrightarrow{BC}|h$，得：

$$h=|\overrightarrow{AB}\times\overrightarrow{AC}|/|\overrightarrow{BC}|=\frac{\sqrt{6^2+6^2+3^3}}{\sqrt{1+(-2)^2+2^2}}=3$$

1-11 若 $\boldsymbol{F}=\boldsymbol{a}_x x$，沿下列三条路径分别计算 $\int_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}$：(1)在 xy 平面沿 x 轴从 $x=0$ 到 $x=1$；(2)沿半径为 1 从 $\varphi=0$ 到 $\varphi=\pi/2$ 的圆弧；(3)沿 y 轴从 $y=1$ 到 $y=0$。

【解】 $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}=\boldsymbol{a}_x x\cdot(\boldsymbol{a}_x\mathrm{d}x+\boldsymbol{a}_y\mathrm{d}y+\boldsymbol{a}_z\mathrm{d}z)=x\mathrm{d}x$

(1) $\int_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}=\int_0^1 x\mathrm{d}x=\frac{x^2}{2}\Big|_0^1=\frac{1}{2}$

(2) $x=1\times\cos\varphi=\cos\varphi$，$\mathrm{d}x=-\sin\varphi\mathrm{d}\varphi$

$$\int_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}=\int_0^{\pi/2}-\cos\varphi\sin\varphi\mathrm{d}\varphi=\frac{1}{2}\cos^2\varphi\Big|_0^{\pi/2}=-\frac{1}{2}$$

(3) 沿 y 轴即有 $x=0$，$\mathrm{d}x=0$，因此 $\int_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}=0$

1-12 若 $\boldsymbol{A}=\boldsymbol{a}_x z+\boldsymbol{a}_y x-\boldsymbol{a}_z 3y^2 z$，求通量 $\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$。其中 S 为圆柱面 $x^2+y^2=16$ 在第一象限中与 $x=0$、$y=0$ 及 $z=0$ 和 $z=5$ 平面所围成的闭合面。

【解】 闭合面由 $x=0$、$y=0$ 及 $z=0$、$z=5$ 平面及 $x^2+y^2=16$ 柱面构成，其中 $x=0$ 平面的法线方向为 $-\boldsymbol{a}_x$，对应的面积分为：

$$\int_{S_x}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=-\int_{S_x}z\mathrm{d}S_x=-\int_0^5\int_0^4 z\mathrm{d}y\mathrm{d}z=-y\Big|_0^4\frac{1}{2}z^2\Big|_0^5=-50$$

$y=0$ 平面的法线方向为 $-\boldsymbol{a}_y$，对应的面积分为：

$$\int_{S_y}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=-\int_{S_y}x\mathrm{d}S_y=-\int_0^5\int_0^4 x\mathrm{d}x\mathrm{d}z=-\frac{1}{2}x^2\Big|_0^4 z\Big|_0^5=-40$$

平面 $z=0$ 的法线方向为 $-\boldsymbol{a}_z$，对应的面积分为：

$$\int_{S_z}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=\int_{S_z}3y^2z\Big|_{z=0}\mathrm{d}S_z=0$$

平面 $z=5$ 的法线方向为 $\boldsymbol{a}_z$，对应的面积分为：

$$\int_{S_z}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}=-\int_{S_z}3y^2z\Big|_{z=5}\mathrm{d}S_z=-15\int_0^{\pi/2}\int_0^4(\rho\sin\varphi)^2\rho\mathrm{d}\rho\mathrm{d}\varphi=-240\pi$$

对柱面 $x^2+y^2=16$ 的积分转化为圆柱坐标较为方便，即 $\rho=4$，法线方向为 $\boldsymbol{a}_\rho$，$x=\rho\cos\varphi=4\cos\varphi$，$y=\rho\sin\varphi=4\sin\varphi$，$\mathrm{d}S_\rho=\rho\mathrm{d}\varphi\mathrm{d}z=4\mathrm{d}\varphi\mathrm{d}z$，对应的面积分为：

$$\int_{S_\rho}\boldsymbol{A}\cdot\boldsymbol{a}_\rho\mathrm{d}S_\rho=\int_{S_\rho}(\boldsymbol{a}_x\cdot\boldsymbol{a}_\rho z+\boldsymbol{a}_y\cdot\boldsymbol{a}_\rho x-\boldsymbol{a}_z\cdot\boldsymbol{a}_\rho 3y^2z)\mathrm{d}S_\rho$$

$$= \int_{S_\rho} (z\cos\varphi + x\sin\varphi)\mathrm{d}S_\rho = \int_0^5\int_0^{\pi/2}(z\cos\varphi + 4\cos\varphi\sin\varphi)4\mathrm{d}\varphi\mathrm{d}z = 90$$

于是得到矢量场 **A** 的闭合面积分为：

$$\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = -50 - 40 - 240\pi + 90 = -240\pi$$

【另解】　利用散度定理 $\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \int_\tau \nabla\cdot\boldsymbol{A}\mathrm{d}\tau$

$$\nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = -3y^2$$

$$\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \int_\tau \nabla\cdot\boldsymbol{A}\mathrm{d}\tau = -\int_0^5\int_0^{\pi/2}\int_0^4 3\,(\rho\sin^2\varphi)^2\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z = -240\pi$$

1-13　球心在原点、半径为 a 的球内充满体密度为 $\rho_f = kr^2$（k 为常数）的电荷。求半径为 $a/2$ 的球面所包围的电荷量和总电量。

【解】

$$Q_{a/2} = \int_\tau \rho_f\mathrm{d}\tau = \int_0^{a/2} kr^2 4\pi r^2\mathrm{d}r = 4\pi k\frac{1}{5}r^5\Big|_0^{a/2} = \frac{\pi k a^5}{40}$$

$$Q = \int_\tau \rho_f\mathrm{d}\tau = \int_0^{a} kr^2 4\pi r^2\mathrm{d}r = 4\pi k\frac{1}{5}r^5\Big|_0^{a} = \frac{4\pi k a^5}{5}$$

1-14　证明：如果 $\boldsymbol{P}\cdot\boldsymbol{A} = \boldsymbol{P}\cdot\boldsymbol{B}$，且 $\boldsymbol{P}\times\boldsymbol{A} = \boldsymbol{P}\times\boldsymbol{B}$，则 $\boldsymbol{A} = \boldsymbol{B}$。

证明：由 $\boldsymbol{P}\cdot\boldsymbol{A} = \boldsymbol{P}\cdot\boldsymbol{B}$ 可得：

$$\boldsymbol{P}\cdot\boldsymbol{A} - \boldsymbol{P}\cdot\boldsymbol{B} = \boldsymbol{P}\cdot(\boldsymbol{A} - \boldsymbol{B}) = 0$$

因此，$\boldsymbol{P}\perp(\boldsymbol{A} - \boldsymbol{B})$　　(1)

又由 $\boldsymbol{P}\times\boldsymbol{A} = \boldsymbol{P}\times\boldsymbol{B}$ 可得：

$$\boldsymbol{P}\times\boldsymbol{A} - \boldsymbol{P}\times\boldsymbol{B} = \boldsymbol{P}\times(\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{0}$$

因此，$\boldsymbol{P}/\!/(\boldsymbol{A} - \boldsymbol{B})$　　(2)

(1)、(2) 两式若同时满足，只有 $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{0}$，即 $\boldsymbol{A} = \boldsymbol{B}$。

1-15　根据算符 ∇ 的矢量特性，推导下列公式：

(1) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$；

(2) $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H})$。

【解】　矢量恒等式与坐标系无关，因此在直角坐标系中推导的结论是普适的。

在直角坐标系中 $\nabla = \boldsymbol{a}_x\frac{\partial}{\partial x} + \boldsymbol{a}_y\frac{\partial}{\partial y} + \boldsymbol{a}_z\frac{\partial}{\partial z}$

(1) 设 $\boldsymbol{A} = \boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z$，$\boldsymbol{B} = \boldsymbol{a}_x B_x + \boldsymbol{a}_y B_y + \boldsymbol{a}_z B_z$

则

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\nabla\times\boldsymbol{A} = \boldsymbol{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \boldsymbol{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \boldsymbol{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

分别计算等式右边各项：

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{A})=\boldsymbol{a}_x\left(B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right)+$$

$$\boldsymbol{a}_y\left(B_z\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-B_x\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right)+\boldsymbol{a}_z\left(B_x\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)-B_y\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\right)$$

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)(\boldsymbol{a}_xA_x+\boldsymbol{a}_yA_y+\boldsymbol{a}_zA_z)$$

$$=\boldsymbol{a}_x\left(B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}\right)+\boldsymbol{a}_y\left(B_x\frac{\partial A_y}{\partial x}+B_y\frac{\partial A_y}{\partial y}+B_z\frac{\partial A_y}{\partial z}\right)+$$

$$\boldsymbol{a}_z\left(B_x\frac{\partial A_z}{\partial x}+B_y\frac{\partial A_z}{\partial y}+B_z\frac{\partial A_z}{\partial z}\right)$$

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{B})=\boldsymbol{a}_x\left(A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\right)+$$

$$\boldsymbol{a}_y\left(A_z\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)-A_x\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right)+$$

$$\boldsymbol{a}_z\left(A_x\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)-A_y\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\right)$$

$$(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}=\boldsymbol{a}_x\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)+\boldsymbol{a}_y\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right)+$$

$$\boldsymbol{a}_z\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)$$

把以上四式相加,得到:

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$$

$$=\boldsymbol{a}_x\left(A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_y}{\partial x}+B_y\frac{\partial A_y}{\partial x}+A_z\frac{\partial B_z}{\partial x}+B_z\frac{\partial A_z}{\partial x}\right)+$$

$$\boldsymbol{a}_y\left(A_x\frac{\partial B_x}{\partial y}+B_x\frac{\partial A_x}{\partial y}+A_y\frac{\partial B_y}{\partial y}+B_y\frac{\partial A_y}{\partial y}+A_z\frac{\partial B_z}{\partial y}+B_z\frac{\partial A_z}{\partial y}\right)+$$

$$\boldsymbol{a}_z\left(A_x\frac{\partial B_x}{\partial z}+B_x\frac{\partial A_x}{\partial z}+A_y\frac{\partial B_y}{\partial z}+B_y\frac{\partial A_y}{\partial z}+A_z\frac{\partial B_z}{\partial z}+B_z\frac{\partial A_z}{\partial z}\right)$$

$$=\boldsymbol{a}_x\frac{\partial}{\partial x}(A_xB_x+A_yB_y+A_zB_z)+\boldsymbol{a}_y\frac{\partial}{\partial y}(A_xB_x+A_yB_y+A_zB_z)+$$

$$\boldsymbol{a}_z\frac{\partial}{\partial z}(A_xB_x+A_yB_y+A_zB_z)$$

$$=\nabla(\boldsymbol{A}\cdot\boldsymbol{B})$$

(2) 设 $\boldsymbol{E}=\boldsymbol{a}_xE_x+\boldsymbol{a}_yE_y+\boldsymbol{a}_zE_z,\boldsymbol{H}=\boldsymbol{a}_xH_x+\boldsymbol{a}_yH_y+\boldsymbol{a}_zH_z$

则 $$\boldsymbol{E}\times\boldsymbol{H}=\boldsymbol{a}_x(E_yH_z-E_zH_y)+\boldsymbol{a}_y(E_zH_x-E_xH_z)+\boldsymbol{a}_z(E_xH_y-E_yH_x)$$

$$\boldsymbol{H}\cdot(\nabla\times\boldsymbol{E})=H_x\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+H_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+H_z\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

$$-\boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}) = -E_x\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)-E_y\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)-E_z\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)$$

将以上两式相加,得到:

$$\boldsymbol{H}\cdot(\nabla\times\boldsymbol{E})-\boldsymbol{E}\cdot(\nabla\times\boldsymbol{H}) = \frac{\partial}{\partial x}(E_yH_z-E_zH_y)+$$

$$\frac{\partial}{\partial y}(E_zH_x-E_xH_z)+\frac{\partial}{\partial z}(E_xH_y-E_yH_x)$$

$$=\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})$$

1-16　证明在圆柱坐标系中,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{\rho^2}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial\varphi^2}\right]$$

证明: 由 $\rho=\sqrt{x^2+y^2}$, $\varphi=\arctan\frac{y}{x}$, 及 $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ 可得:

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial x}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial\rho}\frac{x}{\rho}-\frac{\partial}{\partial\varphi}\frac{y}{\rho^2}=\frac{\partial}{\partial\rho}\cos\varphi-\frac{\partial}{\partial\varphi}\frac{\sin\varphi}{\rho}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial\rho}\frac{y}{\rho}+\frac{\partial}{\partial\varphi}\frac{x}{\rho^2}=\frac{\partial}{\partial\rho}\sin\varphi+\frac{\partial}{\partial\varphi}\frac{\cos\varphi}{\rho}$$

$$\frac{\partial^2}{\partial x^2}=\frac{\partial(\partial/\partial x)}{\partial\rho}\frac{\partial\rho}{\partial x}+\frac{\partial(\partial/\partial x)}{\partial\varphi}\frac{\partial\varphi}{\partial x}$$

$$=\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}\cos\varphi-\frac{\partial}{\partial\varphi}\frac{\sin\varphi}{\rho}\right)\cos\varphi-\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\rho}\cos\varphi-\frac{\partial}{\partial\varphi}\frac{\sin\varphi}{\rho}\right)\frac{\sin\varphi}{\rho}$$

$$=\left(\frac{\partial^2}{\partial\rho^2}\cos\varphi-\frac{\partial^2}{\partial\rho\,\partial\varphi}\frac{\sin\varphi}{\rho}+\frac{\partial}{\partial\varphi}\frac{\sin\varphi}{\rho^2}\right)\cos\varphi-$$

$$\left(\frac{\partial^2}{\partial\varphi\,\partial\rho}\cos\varphi-\frac{\partial}{\partial\rho}\sin\varphi-\frac{\partial^2}{\partial\varphi^2}\frac{\sin\varphi}{\rho}-\frac{\partial}{\partial\varphi}\frac{\cos\varphi}{\rho}\right)\frac{\sin\varphi}{\rho}$$

$$\frac{\partial^2}{\partial y^2}=\frac{\partial(\partial/\partial y)}{\partial\rho}\frac{\partial\rho}{\partial y}+\frac{\partial(\partial/\partial y)}{\partial\varphi}\frac{\partial\varphi}{\partial y}$$

$$=\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}\sin\varphi+\frac{\partial}{\partial\varphi}\frac{\cos\varphi}{\rho}\right)\sin\varphi+\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\rho}\sin\varphi+\frac{\partial}{\partial\varphi}\frac{\cos\varphi}{\rho}\right)\frac{\cos\varphi}{\rho}$$

$$=\left(\frac{\partial^2}{\partial\rho^2}\sin\varphi+\frac{\partial^2}{\partial\rho\,\partial\varphi}\frac{\cos\varphi}{\rho}-\frac{\partial}{\partial\varphi}\frac{\cos\varphi}{\rho^2}\right)\sin\varphi+$$

$$\left(\frac{\partial^2}{\partial\varphi\,\partial\rho}\sin\varphi+\frac{\partial}{\partial\rho}\cos\varphi+\frac{\partial^2}{\partial\varphi^2}\frac{\cos\varphi}{\rho}-\frac{\partial}{\partial\varphi}\frac{\sin\varphi}{\rho}\right)\frac{\cos\varphi}{\rho}$$

于是

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\varphi^2}=\frac{1}{\rho^2}\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial\varphi^2}\right]$$

1-17　求 $f=3x^2y-xy+z^2$ 在点 $(1,-1,1)$ 沿曲线 C 的 x 增加一方的方向导数。已知 C 的曲线方程为:

$$\begin{cases} y = -x^2 \\ z = x^3 \end{cases}$$

【解】 $$\nabla f\big|_{(1,-1,1)} = \left(\boldsymbol{a}_x \frac{\partial f}{\partial x} + \boldsymbol{a}_y \frac{\partial f}{\partial y} + \boldsymbol{a}_z \frac{\partial f}{\partial z}\right)\bigg|_{(1,-1,1)}$$

$$= [\boldsymbol{a}_x(6xy - y) + \boldsymbol{a}_y(3x^2 - x) + \boldsymbol{a}_z 2z]\big|_{(1,-1,1)} = -\boldsymbol{a}_x 5 + \boldsymbol{a}_y 2 + 2\,\boldsymbol{a}_z$$

曲线 C 的方向：

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{a}_x \mathrm{d}x + \boldsymbol{a}_y \mathrm{d}y + \boldsymbol{a}_z \mathrm{d}z = (\boldsymbol{a}_x - \boldsymbol{a}_y 2x + \boldsymbol{a}_z 3x^2)\mathrm{d}x$$

$$\boldsymbol{a}_l\bigg|_{(1,-1,1)} = \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}l}\bigg|_{(1,-1,1)} = (\boldsymbol{a}_x - \boldsymbol{a}_y 2 + \boldsymbol{a}_z 3)/\sqrt{14}$$

函数 f 在点 $(1,-1,1)$ 的方向导数为：

$$\frac{\mathrm{d}f}{\mathrm{d}l} = \nabla f \cdot \boldsymbol{a}_l\bigg|_{(1,-1,1)} = (-5\times 1 - 2\times 2 + 2\times 3)/\sqrt{14} = -3/\sqrt{14}$$

1-18 已知二维标量场 $f = y^2 - x$。(1)问 f 的等值面是何种曲面？并在 xy 平面上画出 f 的等值线族；(2)求 ∇f；(3)任取一个回路 C，计算 $\oint_C \nabla f \cdot \mathrm{d}\boldsymbol{l}$。

【解】 (1)令 $f = y^2 - x = c$（c 为常数），则有：

$$y^2 = x + c$$

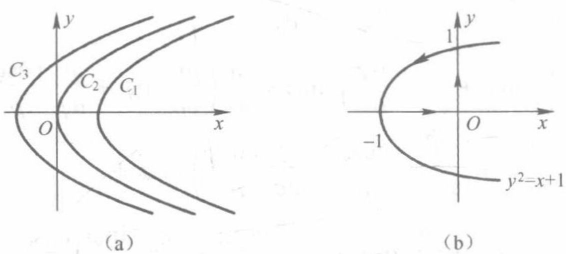
即 f 的等值面是一族抛物柱面，如题 1-18 图(a)所示。

(2) $$\nabla f = \boldsymbol{a}_x \frac{\partial f}{\partial x} + \boldsymbol{a}_y \frac{\partial f}{\partial y} + \boldsymbol{a}_z \frac{\partial f}{\partial z} = -\boldsymbol{a}_x + \boldsymbol{a}_y 2y$$

(3) 设 $c = 1$，取 $y^2 = x + 1$ 与两坐标轴围成的回路，回路绕向沿逆时针，如题 1-18(b)所示，则：

$$\oint_C \nabla f \cdot \mathrm{d}\boldsymbol{l} = \oint_C -\mathrm{d}x + 2y\mathrm{d}y$$

$$= \int_{-1}^{0} -\mathrm{d}x + \int_0^1 2y\mathrm{d}y + \int_1^0 -\mathrm{d}(y^2 - 1) + 2y\mathrm{d}y = -x\big|_{-1}^{0} + y^2\big|_0^1 = 0$$



题 1-18 图

1-19 试求 $\oint_S \boldsymbol{a}_r 3\sin\theta \cdot \mathrm{d}\boldsymbol{S}$，其中 S 为球心位于原点、半径为 5 的球面。

【解】 $$\oint_S \boldsymbol{a}_r 3\sin\theta \cdot \mathrm{d}\boldsymbol{S} = \oint_S 3\sin\theta \mathrm{d}S_r$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\sin\theta 5^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = 75 \times 2\pi \int_0^{\pi} \sin^2\theta \mathrm{d}\theta = 75\pi^2$$

1-20　若矢量 $\boldsymbol{A} = \boldsymbol{a}_r \dfrac{\cos^2\varphi}{r^3}$，$1 \leqslant r \leqslant 2$，试求 $\int_\tau \nabla \cdot \boldsymbol{A} \mathrm{d}\tau$，其中 τ 为 $\boldsymbol{A}$ 所在区域。

【解】　$\nabla \cdot \boldsymbol{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\cos^2\varphi}{r^3}\right) = -\dfrac{1}{r^4}\cos^2\varphi$

$$\int_\tau \nabla \cdot \boldsymbol{A} \mathrm{d}\tau = -\int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\frac{1}{r^4}\cos^2\varphi\right) r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\varphi$$

$$= \frac{1}{r}\Big|_1^2 (-\cos\theta)\Big|_0^{\pi} \left(\frac{1}{2}\varphi + \frac{1}{4}\sin 2\varphi\right)\Big|_0^{2\pi} = -\pi$$

【另解】　利用散度定理 $\int_\tau \nabla \cdot \boldsymbol{A} \mathrm{d}\tau = \oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}$，但要注意做面积分时内外球面的法线方向相反，即外球面法向为 $\boldsymbol{a}_r$，内球面的法向为 $-\boldsymbol{a}_r$。

$$\int_\tau \nabla \cdot \boldsymbol{A} \mathrm{d}\tau = \oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_{S_r} \boldsymbol{A}_r \mathrm{d}\boldsymbol{S}_r$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\cos^2\varphi}{2^3} 2^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi - \int_0^{2\pi}\int_0^{\pi} \frac{\cos^2\varphi}{1^3} 1^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = -\frac{1}{2}\left(\frac{1}{2}\varphi + \frac{1}{4}\sin 2\varphi\right)\Big|_0^{2\pi} (-\cos\theta)\Big|_0^{\pi} = -\pi$$

1-21　应用斯托克斯定理证明 $\int_S \nabla \Psi \times \mathrm{d}\boldsymbol{S} = -\oint_C \Psi \mathrm{d}\boldsymbol{l}$。

【解】　设矢量场 $\boldsymbol{A} = \Psi \boldsymbol{C}$，其中 Ψ 是标量函数，$\boldsymbol{C}$ 是任意常矢量，则：

$$\nabla \times \boldsymbol{A} = \nabla \times (\Psi \boldsymbol{C}) = \nabla \Psi \times \boldsymbol{C} + \Psi \nabla \times \boldsymbol{C} = \nabla \Psi \times \boldsymbol{C}$$

代入斯托克斯定理 $\int_S \nabla \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$ 中，可得：

$$\int_S \nabla \Psi \times \boldsymbol{C} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \Psi \boldsymbol{C} \cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{C} \cdot \oint_C \Psi \mathrm{d}\boldsymbol{l}$$

利用矢量恒等式 $(\boldsymbol{X} \times \boldsymbol{Y}) \cdot \boldsymbol{Z} = (\boldsymbol{Z} \times \boldsymbol{X}) \cdot \boldsymbol{Y} = (-\boldsymbol{X} \times \boldsymbol{Z}) \cdot \boldsymbol{Y}$，上式可写为：

$$-\boldsymbol{C} \cdot \int_S \nabla \Psi \times \mathrm{d}\boldsymbol{S} = \boldsymbol{C} \cdot \oint_C \Psi \mathrm{d}\boldsymbol{l}$$

由于 $\boldsymbol{C}$ 是任意常矢量，因此有：

$$\int_S \nabla \Psi \times \mathrm{d}\boldsymbol{S} = -\oint_C \Psi \mathrm{d}\boldsymbol{l}$$

1-22　在由坐标面 $\rho = 5$，$z = 0$，$z = 2$ 围成的圆柱形区域中，对矢量 $\boldsymbol{A} = \boldsymbol{a}_\rho \rho^2 + \boldsymbol{a}_z 2z$ 验证散度定理。

【解】　矢量 $\boldsymbol{A}$ 只有 A_ρ 和 A_z 分量，因此有：

$$\oint_S \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_S A_\rho \mathrm{d}S_\rho + A_z \mathrm{d}\boldsymbol{S}_z$$

$$= \int_{S_{\rho=5}} \rho^2 \rho \mathrm{d}z\mathrm{d}\varphi - \int_{S_{z=0}} 2z\rho\mathrm{d}\rho\mathrm{d}\varphi + \int_{S_{z=2}} 2z\rho\mathrm{d}\rho\mathrm{d}\varphi$$

$$= \int_0^{2\pi}\int_0^2 5^2 5\mathrm{d}z\mathrm{d}\varphi + \int_0^{2\pi}\int_0^5 2\times 2\rho\mathrm{d}\rho\mathrm{d}\varphi = 600\pi$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{\partial A_z}{\partial z} = 3\rho + 2$$

$$\int_\tau \nabla\cdot\boldsymbol{A}\mathrm{d}\tau = \int_0^{2\pi}\int_0^2\int_0^5 (3\rho+2)\rho\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi = 600\pi$$

所以
$$\oint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \int_\tau \nabla\cdot\boldsymbol{A}\mathrm{d}\tau$$

1-23　在矢量场 $\boldsymbol{A} = -\boldsymbol{a}_x y + \boldsymbol{a}_y x$ 中有一矩形回路 C:(0,0)→(3,0)→(3,4)→(0,4)→(0,0)。对回路 C 及其围成的矩形面积 S 验证斯托克斯定理。

【解】　回路 C 包括四段:$y=0$,x 从 0 到 3; $x=3$,y 从 0 到 4;$y=4$,x 从 3 到 0;$x=0$,y 从 4 到 0。于是

$$\oint_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \oint_C -y\mathrm{d}x + x\mathrm{d}y$$

$$= \int_0^3 -0\mathrm{d}x + \int_0^4 3\mathrm{d}y + \int_3^0 -4\mathrm{d}x + \int_3^0 0\mathrm{d}y = 24$$

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \boldsymbol{a}_z 2$$

$$\int_S \nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = 2S = 2\times 3\times 4 = 24$$

所以
$$\oint_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \int_S \nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$$

1-24　有三个矢量场:

$$\boldsymbol{A} = \boldsymbol{a}_r\sin\theta\cos\varphi + \boldsymbol{a}_\theta\cos\theta\cos\varphi - \boldsymbol{a}_\varphi\sin\varphi$$

$$\boldsymbol{B} = \boldsymbol{a}_\rho z^2\sin\varphi + \boldsymbol{a}_\varphi z^2\cos\varphi + \boldsymbol{a}_z 2\rho z\sin\varphi$$

$$\boldsymbol{C} = \boldsymbol{a}_x(3y^2 - 2x) + \boldsymbol{a}_y x^2 + \boldsymbol{a}_z 2z$$

问:其中哪些场可以表示为一个标量场的梯度场?哪些场可以表示为一个矢量场的旋度场?求出这些矢量场的源分布。

【解】　$$\nabla\times\boldsymbol{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\sin\theta\,\boldsymbol{a}_\varphi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \end{vmatrix} = 0$$

$$\nabla\times\boldsymbol{B}=\frac{1}{\rho}\begin{vmatrix}\boldsymbol{a}_\rho & \rho\,\boldsymbol{a}_\varphi & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z}\\ z^2\sin\varphi & \rho z^2\cos\varphi & 2\rho z\sin\varphi\end{vmatrix}=0$$

$$\nabla\times\boldsymbol{C}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3y^2-2x & x^2 & 2z\end{vmatrix}=\boldsymbol{a}_z(2x-6y)$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sin\theta\cos\varphi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\cos\varphi)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(-\sin\theta)=0$$

$$\nabla\cdot\boldsymbol{B}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho z^2\sin\varphi)+\frac{1}{\rho}\frac{\partial}{\partial\varphi}(z^2\cos\varphi)+\frac{\partial}{\partial z}(2\rho z\sin\varphi)=2\rho\sin\varphi$$

$$\nabla\cdot\boldsymbol{C}=\frac{\partial}{\partial x}(3y^2-2x)+\frac{\partial}{\partial y}(x^2)+\frac{\partial}{\partial z}(2z)=0$$

以上结果表明矢量场 $\boldsymbol{A}$、$\boldsymbol{B}$ 可表示为一个标量场的梯度，$\boldsymbol{A}$、$\boldsymbol{C}$ 可表示为一个矢量场的旋度，但矢量场 $\boldsymbol{A}$ 既无散也无旋，并无物理意义。

1-25 已知无限大空间矢量场 $\nabla\cdot\boldsymbol{F}=q\delta(\boldsymbol{r})$，$\nabla\times\boldsymbol{F}=0$，试求该矢量场。

【解】 已知矢量场的散度和旋度时，根据亥姆霍兹定理，矢量场可求得唯一解：

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r})&=-\frac{1}{4\pi}\nabla\int_\tau\frac{\nabla\cdot\boldsymbol{F}(\boldsymbol{r}')}{r}\mathrm{d}\tau'+\frac{1}{4\pi}\nabla\times\int_\tau\frac{\nabla\times\boldsymbol{F}(\boldsymbol{r}')}{r}\mathrm{d}\tau'\\&=-\frac{1}{4\pi}\nabla\frac{q}{r}+0=\boldsymbol{a}_r\frac{q}{4\pi r^2}\end{aligned}$$