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ANALYSIS, MANIFOLDS AND PHYSICS

Part: II

Revised and Enlarged Edition

YVONNE CHOQUET-BRUHAT, C. DeWITT-MORETTE

分析流形和物理学

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Part II

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Part II

PREFACE TO THE SECOND EDITION

Twelve problems have been added to the first edition; four of them are supplements to problems in the first edition. The others deal with issues that have become important, since the first edition of Volume II, in recent developments of various areas of physics. All the problems have their foundations in Volume I of the 2-Volume set *Analysis, Manifolds, and Physics.*

It would have been prohibitively expensive to insert the new problems at their respective places. They are grouped together at the end of this volume, their logical place is indicated by a number in parenthesis following the title.

The new problems are:

- "The isomorphism $\mathbb{H} \oplus \mathbb{H} \simeq M_4(\mathbb{R})$. A supplement to Problem I.4 and I.3 (I.17)." Its logical place is the seventeenth problem of Chapter I.
- The problem "Lie derivative of spinor fields (III.15)" belongs to Chapter III.
- "Poisson–Lie groups, Lie bialgebras, and the generalized classical Yang–Baxter equation (IV.14)" has been contributed by Carlos Moreno and Luis Valero. It belongs to Chapter IV.

  Additions to Chapter V on Riemannian and Kählerian manifolds include:
- "Volume of the sphere S^n. A supplement to Problem V.4 (V.15)"
- "Teichmuller spaces (V.16)"
- "Yamabe property on compact manifolds (V.17)"

  To Chapter V bis on Connections are added:
- "The Euler class. A supplement to Problem V bis 6 (V bis 13)"
- "Formula of Laplacians at a point of the frame bundle (V bis 14)"
- "The Berry and Aharanov–Anandan phases (V bis 15)" based on notes by Ali Mostafazadeh.

  To Chapter VI on Distributions:
- "A density theorem. A supplement to Problem VI.6 on 'spaces $H_{s,\delta}(\mathbb{R}^m)$' (VI.17)"
- Tensor distributions on submanifolds, multiple layers, and shocks (VI.18)"
- "Discrete Boltzman equation (VI.19)"

A fair number of misprints have been corrected. An updated list of errata for Volume I is included.

Naturally more problems are on our drawing boards. We would like to think of them as contributions to a third edition.

Most of the new problems were completed during a visit of Y. Choquet-Bruhat to the Center for Relativity of the University of Texas, made possible by the Jane and Roland Blumberg Centennial Professorship in Physics held by C. DeWitt-Morette. Help and comments from M. Berg, M. Blau, M. Godina, S. Gutt, M. Smith, R. Stora, X. Wu-Morrow and A. Wurm are gratefully acknowledged.

PREFACE

This book is a companion volume to our first book, *Analysis, Manifolds and Physics* (Revised Edition 1982). In the context of applications of current interest in physics, we develop concepts and theorems, and present topics closely related to those of the first book. The first book is not necessary to the reader interested in Chapters I–V bis and already familiar with differential geometry nor to the reader interested in Chapter VI and already familiar with distribution theory. The first book emphasizes basics; the second, recent applications.

Applications are the lifeblood of concepts and theorems. They answer questions and raise questions. We have used them to provide motivation for concepts and to present new subjects that are still in the developmental stage. We have presented the applications in the forms of the problems with solutions in order to stress the questions we wish to answer and the fundamental ideas underlying applications. The reader may also wish to read only the questions and work out for himself the answers, one of the best ways to learn how to use a new tool. Occasionally we had to give a longer-than-usual introduction before presenting the questions. The organization of questions and answers does not follow a rigid scheme but is adapted to each problem.

This book is coordinated with the first one as follows:

1. The chapter headings are the same – but in this book, there is no Chapter VII devoted to infinite dimensional manifolds per se. Instead, the infinite dimensional applications are treated together with the corresponding finite dimensional ones and can be found throughout the book.
2. The subheadings of the first book have not been reproduced in the second one because applications often use properties from several sections of a chapter. They may even, occasionally, use properties from subsequent chapters and have been placed according to their dominant contribution.
3. **Page numbers in parentheses refer to the first book.** References to other problems in the present book are indicated [Problem Chapter Number First Word of Title].

The choice of problems was guided by recent applications of differential geometry to fundamental problems of physics, as well as by our personal interests. It is, in part, arbitrary and limited by time, space, and our desire to bring this project to a close.

The references are not to be construed as an exhaustive bibliography; they are mainly those that we used while we were preparing a problem or that we came across shortly after its completion.

The book has been enriched by contributions of Charles Doering, Harold Grosse, B. Kent Harrison, N.H Ibragimov, and Carlos Moreno, and collaborations with Ioannis Bakas, Steven Carlip, Gary Hamrick, Humberto La Roche and Gary Sammelmann. Discussions with S. Blau, M. Dubois-Violette, S.G. Low, L.C. Shepley, R. Stora, A. H. Taub, J. Tits and Jahja Trisnadi are gratefully acknowledged.

The manuscript has been prepared by Ms. Serot Almeras, Peggy Caffrey, Jan Duffy and Elizabeth Stepherd.

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CONVENTIONS

(1) $\{f_n\}_{\mathbb{N}} := \{f_n : n \in \mathbb{N}\}$.

(2) Commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & h \searrow \bullet \swarrow g & \\ & z & \end{array} \Leftrightarrow \begin{cases} f : x \to y, \; g : y \to z \\ h = g \circ f \end{cases}.$$

(3) Integer part: if $d/2 = 3.5$, then $[d/2] = 3$.

(4) $A\backslash B$ and A/B sometimes mean left and right coset, respectively; but usage varies and is determined in each context.

(5) Exterior product, exterior derivative, interior product

$$(\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\Pi} (\operatorname{sign} \Pi) \Pi [\alpha(v_1, \ldots, v_p) \times \beta(v_{p+1}, \ldots, v_{p+q})],$$

$$(\alpha \,\bar{\wedge}\, \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\Pi} (\operatorname{sign} \Pi) \Pi [\alpha(v_1, \ldots, v_p) \beta \times (v_{p+1}, \ldots, v_{p+q})].$$

When operating on a p-form $\bar{d} = d/(p+1)$ and $\bar{\iota}_v = p\iota_v$. Note that Kobayashi and Nomizu (Vol. I, p. 35) use what we call $\bar{\wedge}$.

(6) Riemann tensor, Ricci tensor

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\lambda = R_{\alpha\beta}{}^\lambda{}_\mu v^\mu,$$

i.e.

$$R_{\alpha\beta}{}^\lambda{}_\mu = \partial_\alpha \Gamma_\beta{}^\lambda{}_\mu - \partial_\beta \Gamma_\alpha{}^\lambda{}_\mu + \Gamma_\alpha{}^\rho{}_\mu \Gamma_\beta{}^\lambda{}_\rho - \Gamma_\beta{}^\rho{}_\mu \Gamma_\alpha{}^\lambda{}_\rho.$$

$$R_{\beta\mu} := R_{\alpha\beta}{}^\alpha{}_\mu.$$

These conventions agree with Misner, Thorne, and Wheeler and differ from those of our first book *Analysis, Manifolds and Physics*.

(7) The Dirac representation of the gamma matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+, +, +, -)$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}.$$

Majorana representation of the gamma matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \qquad \eta_{\mu\nu} = \mathrm{diag}(+, +, +, -)$$

$$\gamma_1' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma_3' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_4' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that in Vol. I, p. 176, we give the Dirac representation of the gamma matrices for $\eta_{\mu\nu} = \mathrm{diag}(+, -, -, -)$.

CONTENTS

I. REVIEW OF FUNDAMENTAL NOTIONS OF ANALYSIS

1. GRADED ALGEBRAS

For applications and references see, for instance, Problems II 1, Supersmooth mappings and III 14, Graded bundles.

A $\mathbb{Z}_2$ **graded algebra** A is a vector space over the field of real or complex numbers which is the direct sum of two subspaces A_+ (called even) and A_- (called odd)

graded algebra

$$A = A_+ \oplus A_-$$

endowed with an associative and distributive operation, called product, such that

$$A_r A_s = A_{r+s} (\mathrm{mod.}\, 2), \quad r, s = 0, 1, \qquad A_0 = A_+, \qquad A_1 = A_-.$$

A $\mathbb{Z}_2$ graded algebra is called **graded commutative** if any two odd elements anticommute and if even elements commute with all others:

graded commutative

$$ab = (-1)^{d(a)d(b)} ba, \quad a, b \in A$$

where $d(a) = r$ if $a \in A_r$ is the **parity** of a.

parity

We shall consider in this section only graded commutative algebras, so we shall omit the word "commutative".
The algebras we shall use will be endowed with a locally convex Hausdorff topology for which sum and product are continuous operations.
For example, the exterior (Grassmann) algebra over a finite dimensional vector space X (p. 196) is a graded algebra.
A generalization used in physics, which we shall call a (Bryce) **DeWitt algebra** is the algebra B of formal series with a unit e and an infinite number of generators z^I, $I \in \mathbb{N}$, with the usual sum and product laws and the anticommutation property

(Bryce) DeW algebra B

$$z^I z^J = -z^J z^I.$$

An element $a \in B$ is written (notion of convergence is irrelevant)

$$a = \sum_{p \in \mathbb{N}} a(p), \qquad a(p) = \frac{1}{p!} a_{I_1 \ldots I_p} z^{I_1} \ldots z^{I_p}.$$