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分数因子和分数消去图

周思中（Sizhong Zhou） 著

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Preface

Graph theory is one of the branches of modern mathematics which has shown impressive advances in recent years. Graph theory is widely applied in physics, chemistry, biology, network theory, information sciences, computer science and other fields, and so it has attracted a great deal of attention.

Factor theory of graph is an important branch in graph theory. It has extensive applications in various areas, e.g., combinatorial design, network design, circuit layout, scheduling problems, the file transfer problems and so on.

The fractional factor problem in graphs can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

In this book, we mainly discuss the fractional factor problem. This book is divided into five chapters.

In Chapter 1, we show basic terminologies, definitions and graphic parameters.

In Chapter 2, we show some sufficient conditions for the existence of fractional factors in graphs. This chapter is divided into three parts. Firstly, we present some sufficient conditions for graphs to have fractional k-factors in terms of neighborhood, binding number, toughness, minimum degree and independence number, etc. It is shown that these results in this part are sharp. Secondly, we investigate the existence of fractional k-factors including any given edge in graphs, and show two results on fractional k-factors including any given edge, and verify that the results are sharp. Thirdly, we study fractional (g, f)-factors with prescribed properties in graphs. We use connectivity and independence number to obtain a sufficient condition for a graph to have a fractional (g, f)-factor, and this result is best possible in some sense. Furthermore, we obtain a result on a fractional (g, f)-factor including any given k edges in a graph.

In Chapter 3, we discuss a generalization of fractional factors, i.e., fractional deleted graphs from different perspectives, such as degree condition, neighborhood condition and binding number. We present some sufficient conditions related to these parameters for the existence of fractional k-deleted graphs and fractional (g, f)-deleted graphs. Furthermore, it is shown that these results are sharp.

In Chapter 4, we study fractional (k, m)-deleted graphs which are the generalizations of fractional k-factors and fractional k-deleted graphs. We first give a criterion for a graph to be a fractional (k, m)-deleted graph. Then we use the criterion to obtain some graphic parameter (such as minimum degree, neighborhood, toughness and binding number , etc.) conditions for graphs to be fractional (k, m)-deleted graphs. Furthermore, it is shown that the results are best possible in some sense.

In Chapter 5, we focus on some sufficient conditions for graphs to be fractional (g, f, m)-deleted graphs. Our results on fractional (g, f, m)-deleted graphs are an extension of the previous results.

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Sizhong Zhou

Oct. 6th, 2014

Contents

Chapter 1

Terminologies and Graphic Parameters

In this chapter, some basic terminologies, definitions and graphic parameters are given, which will be used throughout this book.

1.1 Basic Terminologies

All graphs considered in this work are finite undirected graphs which have neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called a **spanning subgraph** of G if $V(H) = V(G)$. For $x \in V(G)$, we use $N_G(x)$ to denote the neighborhood of x in G and $d_G(x)$ to denote the degree of x in G. For any $X \subseteq V(G)$, we define

$$N_G(X) = \bigcup_{x \in X} N_G(x).$$

Note that $N_G(x)$ does not contain x, but it may happen that $N_G(X) \supseteq X$. Let S be a subset of $V(G)$. Denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with their incident edges. Denote by $G[S]$ the subgraph of G induced by the vertex set S, i.e., the graph having vertex set S and whose edge set consists of those edges of G incident with two elements of S. Let S and T be two disjoint subsets of $V(G)$, we denote by $E_G(S, T)$ the set of edges with one end in S and the other end in T, and we write

$$e_G(S, T) = |E_G(S, T)|.$$

Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer such that $\lfloor r \rfloor \leq r$. Other definitions and terminologies can be found in [3].

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every $x \in V(G)$. A spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$ is a (g, f)-factor of G. Let $h: E(G) \rightarrow [0,1]$ be a function defined on $E(G)$. If

$$g(x) \leq \sum_{e \ni x} h(e) \leq f(x)$$

holds for every $x \in V(G)$, then we call $G[F_h]$ a **fractional** (g, f)**-factor** of G with indicator function h, where

$$F_h = \{e : e \in E(G),\ h(e) > 0\}.$$

A fractional (f, f)-factor is called simply a **fractional** f**-factor**. If $f(x) = k$, then a fractional f-factor is called a **fractional** k**-factor**. A graph G is called a **fractional** (g, f, m)**-deleted graph** if there exists a fractional (g, f)-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (f, f, m)-deleted graph is simply called a **fractional** (f, m)**-deleted graph**. If $f(x) = k$ for any $x \in V(G)$, then a fractional (f, m)-deleted graph is called a **fractional** (k, m)**-deleted graph**. Set $m = 1$. Then a fractional (g, f, m)-deleted graph is a fractional (g, f)-deleted graph; a fractional (f, m)-deleted graph is simple called a **fractional** f**-deleted graph**; a fractional (k, m)-deleted graph is a fractional k-deleted graph. The basic results on graph factors can be found in [1, 17, 24].

1.2 Graphic Parameters

The graphic parameters play important roles in the research of graph factors and fractional factors, they are used frequently as sufficient conditions for the existence of graph factors and fractional factors. Since verifying graphic parameter conditions are often easier than that of characterizations, and as well the parameter conditions reflect the structures and properties of graphs from different perspectives, it is quite common in graph theory to investigate the links among the parameters. In this section, we show some graphic parameters, such as binding number, toughness, isolated toughness, independence number, connectivity and minimum degree, etc.

The minimum degree of G is denoted by $\delta(G)$, i.e.,

$$\delta(G) = \min\{d_G(x) : x \in V(G)\}.$$

The maximum degree of G is denoted by $\Delta(G)$, i.e.,

$$\Delta(G) = \max\{d_G(x) : x \in V(G)\}.$$

A vertex subset S of G is called **independent** if $G[S]$ has no edges. An independent set S of G is called a **maximum independent set** if G excludes a independent set S' with $|S'| > |S|$. The number of vertices in the maximum independent set S of G is called the **independence number**, and is denoted by $\alpha(G)$.

Let G be a connected graph.

$$\kappa(G) = \min\{|T| : T \subseteq V(G),\ G - T \text{ is disconnected or is a trivial graph}\}$$

is called the **connectivity** of G.

$$\lambda(G) = \min\{|E_G(S, V(G) \setminus S)| : S \subseteq V(G)\}$$

is called the **edge-connectivity** of G.

The binding number was introduced by Woodall[21] and is defined as

$$\text{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G),\ N_G(X) \neq V(G)\right\}.$$

Obviously, for any nonempty subset $S \subseteq V(G)$ with $N_G(S) \neq V(G)$, we have

$$|N_G(S)| \geq \text{bind}(G)|S|.$$

The number of connected components in G is denoted by $\omega(G)$. The toughness $t(G)$ of a connected graph G was first defined by Chvatal in [5] as follows.

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G),\ \omega(G-S) \geq 2\right\},$$

if G is not complete; otherwise, $t(G) = +\infty$.

Enomoto[6] introduced a new parameter $\tau(G)$ which is a slight variation of toughness, but seems better to fit the research of graph factors and fractional factors. For a connected graph G, we define

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)-1} : S \subseteq V(G),\ \omega(G-S) \geq 2\right\},$$

if G is not complete; otherwise, $\tau(G) = +\infty$.

we use $i(G)$ to denote the number of isolated vertices of G. The isolated toughness $I(G)$ of a graph G is defined by Ma and Liu[22] as follows.

$$I(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G),\ i(G-S) \geq 2\right\},$$

if G is not complete; otherwise, $I(G) = +\infty$.

Ma and Liu[15] introduced a new parameter $I'(G)$ which is a slight variation of isolated toughness. For a graph G, we define

$$I'(G) = \min\left\{\frac{|S|}{i(G-S)-1} : S \subseteq V(G),\ i(G-S) \geq 2\right\},$$

if G is not complete; otherwise, $I'(G) = +\infty$.

Chapter 2
Fractional Factors

In this chapter we study fractional k-factors and fractional (g, f)-factors which are natural generalizations of k-factors and (g, f)-factors, and investigate fractional k-factors and fractional (g, f)-factors including any given edges. This chapter is divided into three parts. Firstly, We show some sufficient conditions for graphs to have fractional k-factors. Secondly, we give some results on fractional k-factors including any given edges. Thirdly, we obtain some sufficient conditions for graphs to have fractional (g, f)-factors with prescribed properties.

2.1 Fractional k-Factors

Let k be an integer such that $k \geq 1$. Then a spanning subgraph F of G is called a **k-factor** if $d_F(x) = k$ for all $x \in V(G)$. Let $h:\ E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a **fractional k-factor** of G with indicator function h where

$$F_h = \{e \in E(G):\ h(e) > 0\}.$$

A fractional 1-factor is also called a **fractional perfect matching**. If $h(e) \in \{0, 1\}$ for any $e \in E(G)$, then a fractional k-factor is equivalent to a k-factor, and so a fractional k-factor is a natural generalization of a k-factor.

We shall show a necessary and sufficient condition for a graph to have a fractional k-factor, which is a special case of the fractional (g, f)-factor theorem presented by Anstee[2]. Liu and Zhang[12] showed a simple proof of the theorem.

Theorem 2.1.1 [2, 12] *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of V(G),*

$$\delta_G(S,T) = k|S| - k|T| + d_{G-S}(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq k\}$.

The following theorem obtained by Zhang and Liu[25] is equivalent to Theorem 2.1.1.

Theorem 2.1.2 [25] *Let G be a graph. Then G has a fractional k-factor if and only if for all subset S of V(G),*

$$k|S| - \sum_{j=0}^{k-1}(k-j)p_j(G-S) \geq 0,$$

where $p_j(G-S)$ *denotes the number of vertices in* $G-S$ *with degree* j.

In Theorem 2.1.2, let $k=1$, then we get the following result.

Theorem 2.1.3 [18] *A graph G has a fractional* 1*-factor if and only if*

$$i(G-S) \leq |S|$$

for all $S \subseteq V(G)$.

In [7], Iida and Nishimura gave a neighborhood condition for a graph to have a k-factor.

Theorem 2.1.4 [7] *Let* k *be an integer such that* $k \geq 2$, *and let G be a connected graph of order* n *such that*

$$n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2},$$

kn *is even, and the minimum degree is at least* k. *If G satisfies*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices $x, y \in V(G)$, *then G has a* k*-factor.*

In the following we shall present some results on fractional k-factors by using Theorems 2.1.1–2.1.3.

We first give some neighborhood conditions for the existence of fractional k-factors in graphs.[37, 40, 42]

Theorem 2.1.5 [40] *Let G be a connected graph of order n such that $n \geq 3$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional 1-factor.

Theorem 2.1.6 [40] *Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that*

$$n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2},$$

and the minimum degree $\delta(G) \geq k$. If

$$|N_G(x) \cup N_G(y)| \geq \max\left\{\frac{n}{2}, \frac{1}{2}(n+k-2)\right\}$$

for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

The following lemmas is often applied in the proof of Theorems 2.1.5–2.1.6.

Lemma 2.1.1 [7] *Let k be an integer such that $k \geq 1$. Then*

$$9k - 1 - 4\sqrt{2(k-1)^2 + 2} \begin{cases} > 3k+5, & \text{for } k \geq 4, \\ > 3k+4, & \text{for } k = 3, \\ = 3k+3, & \text{for } k = 2, \\ > 2, & \text{for } k = 1. \end{cases}$$

Lemma 2.1.2 [40] *Let G be a connected graph of order n. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of nonadjacent vertices $x, y \in V(G)$, then

$$\omega(G - S) \le |S| + 1$$

for all $S \subseteq V(G)$ with $|S| \ge 2$.

Proof Suppose that there exists a vertex subset $S \subseteq V(G)$ with $|S| \ge 2$ such that $\omega(G - S) \ge |S| + 2$.

Claim $2 \le |S| \le \dfrac{n-2}{2}$.

Proof If $|S| \ge \frac{n-1}{2}$, then

$$\begin{aligned}\omega(G - S) &\le n - |S| \le n - \frac{n-1}{2} = \frac{n+1}{2} = \frac{n-1}{2} + 1 \\ &\le |S| + 1,\end{aligned}$$

a contradiction. □

In the following, let $C_1, C_2, \cdots, C_\omega$ be the connected components of $G - S$. We have

$$\omega = \omega(G - S) \ge |S| + 2 \ge 4$$

since $|S| \ge 2$. Choose an arbitrary vertex $x_i \in V(C_i)$ $(1 \le i \le \omega(G - S))$. Then $x_i x_j \notin E(G)$ $(i \ne j)$. By the hypothesis of the lemma, then

$$\begin{aligned}\frac{n}{2} &\le |N_G(x_i) \cup N_G(x_j)| \le d_{G-S}(x_i) + d_{G-S}(x_j) + |S| \\ &\le |V(C_i)| - 1 + |V(C_j)| - 1 + |S|,\end{aligned}$$

that is,

$$|V(C_i)| + |V(C_j)| \ge \frac{n+4}{2} - |S|$$

for $i \ne j$. Thus, we get that

$$\begin{aligned}n &= |S| + |V(C_1)| + |V(C_2)| + \cdots + |V(C_\omega)| \\ &= |S| + \frac{1}{2}[2(|V(C_1)| + |V(C_2)| + \cdots + |V(C_\omega)|)] \\ &\ge |S| + \frac{\omega}{2}\Big(\frac{n+4}{2} - |S|\Big) \\ &\ge |S| + \frac{|S|+2}{2}\Big(\frac{n+4}{2} - |S|\Big)\end{aligned}$$

$$= -\frac{1}{2}|S|^2 + \frac{n+4}{4}|S| + \frac{n+4}{2},$$

that is,

$$n \geq -\frac{1}{2}|S|^2 + \frac{n+4}{4}|S| + \frac{n+4}{2}. \tag{2.1.1}$$

Let

$$f(|S|) = -\frac{1}{2}|S|^2 + \frac{n+4}{4}|S| + \frac{n+4}{2}.$$

In fact, the function $f(|S|)$ attains its minimum value at $|S| = 2$ or $|S| = \frac{n-2}{2}$ which follows from $2 \leq |S| \leq \frac{n-2}{2}$, and the minimum value of the function $f(|S|)$ is $\min\{n+2, \frac{5n+2}{4}\}$. According to (2.1.1) and the minimum value of the function $f(|S|)$, we have

$$n > n,$$

which is a contradiction. Completing the proof of the lemma. □

Proof of Theorem 2.1.5 If G is a graph which satisfies the condition of the theorem and there is no fractional 1-factor in G, then by Theorem 2.1.3, there exists a vertex set $S \subseteq V(G)$ such that

$$i(G-S) > |S|. \tag{2.1.2}$$

Clearly, $S \neq \emptyset$ since G is connected. In the following, we assume $|S| \geq 1$. The proof splits into two cases.

Case 1 $|S| = 1$.

Thus, we get that $i(G-S) > |S| = 1$, that is,

$$i(G-S) \geq 2.$$

Let $x, y \in I(G-S)$, then $xy \notin E(G)$ and $d_{G-S}(x) = d_{G-S}(y) = 0$. According to the hypothesis of the theorem, we have

$$\frac{n}{2} \leq |N_G(x) \cup N_G(y)| \leq d_{G-S}(x) + d_{G-S}(y) + |S| = |S| = 1.$$

Therefore, we get $n \leq 2$. This contradicts $n \geq 3$.

Case 2 $|S| \geq 2$.

We have known that $\omega(G-S) \geq i(G-S)$ and it implies that

$$\omega(G-S) \geq |S| + 1$$

by (2.1.2). But $\omega(G-S) \leq |S|+1$ for all $S \subseteq V(G)$ with $|S| \geq 2$ by Lemma 2.1.2. Thus, we have

$$\omega(G-S) = |S| + 1$$

for all $S \subseteq V(G)$ with $|S| \geq 2$. Moreover, we have

$$i(G-S) = |S| + 1$$

and

$$|S| \leq \frac{n-1}{2}. \tag{2.1.3}$$

For any two vertices $x, y \in I(G-S)$, obviously, $xy \notin E(G)$ and $d_{G-S}(x) = d_{G-S}(y) = 0$. Then, we have

$$\frac{n}{2} \leq |N_G(x) \cup N_G(y)| \leq d_{G-S}(x) + d_{G-S}(y) + |S| = |S| \leq \frac{n-1}{2}$$

by the hypothesis of the theorem and (2.1.3). It is a contradiction. Completing the proof of the theorem. □

Proof of Theorem 2.1.6 By Theorem 2.1.5, the result obviously holds for $k = 1$. If $k \geq 2$ and kn is even, then G has a k-factor by Theorem 2.1.4. We have known that a k-factor is a special fractional k-factor. Now we consider the case that k and n are both odd.

If G has no fractional k-factor, then by Theorem 2.1.1, there exists some $S \subseteq V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq -1, \tag{2.1.4}$$

where

$$T = \{x : \ x \in V(G) \setminus S, \ d_{G-S}(x) \leq k\}.$$

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (2.1.4). As k is odd, therefore, $(k-1)$ is even and G has a fractional $(k-1)$-factor by Theorem 2.1.4.

Claim 1 $|T| \geq |S| + 1$.

Proof Since G has a fractional $(k-1)$-factor, then we get

$$(k-1)|S| + d_{G-S}(T') - (k-1)|T'| \geq 0$$

where $T' = \{x : \ x \in V(G) \setminus S, \ d_{G-S}(x) \leq k-1\}$. Moreover,