Andrew J. Majda, Xiaoming Wang

Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows

非线性动力学和统计理论在地球物理流动中的应用

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ANDREW J. MAJDA
New York University
XIAOMING WANG
Florida State University

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Preface

This book is an introduction to the fascinating and important interplay between non-linear dynamics and statistical theories for geophysical flows. The book is designed for a multi-disciplinary audience ranging from beginning graduate students to senior researchers in applied mathematics as well as theoretically inclined graduate students and researchers in atmosphere/ocean science. The approach in this book emphasizes the serendipity between physical phenomena and modern applied mathematics, including rigorous mathematical analysis, qualitative models, and numerical simulations. The book includes more conventional topics for non-linear dynamics applied to geophysical flows, such as long time selective decay, the effect of large-scale forcing, non-linear stability and fluid flow on the sphere, as well as emerging contemporary research topics involving applications of chaotic dynamics, equilibrium statistical mechanics, and information theory. The various competing approaches for equilibrium statistical theories for geophysical flows are compared and contrasted systematically from the viewpoint of modern applied mathematics, including an application for predicting the Great Red Spot of Jupiter in a fashion consistent with the observational record. Novel applications of information theory are utilized to simplify, unify, and compare the equilibrium statistical theories and also to quantify aspects of predictability in non-linear dynamical systems with many degrees of freedom. No previous background in geophysical flows, probability theory, information theory, or equilibrium statistical mechanics is needed to read the text. These topics and related background concepts are all introduced and developed through elementary examples and discussion throughout the text as they arise. The book is also of wider interest to applied mathematicians and other scientists to illustrate how ideas from statistical physics can be applied in novel ways to inhomogeneous large-scale complex non-linear systems.

The material in the book is based on lectures of the first author given at the Courant Institute in 1995, 1997, 2001, and 2004. The first author thanks Professor

Pedro Embid as well as his former Ph.D. students Professor Pete Kramer and Seuyung Shim for their help with early versions of Chapters 1, 2, 3, 4, and 6 of the present book. Joint research work with Professors Richard Kleeman and Bruce Turkington as well as Majdas former Courant post docs, Professors Marcus Grote, Ilya Timofeyev, Rafail Abramov, and Mark DeBattista have been incorporated into the book; their explicit and implicit contributions are acknowledged warmly. The authors acknowledge generous support of the National Science Foundation and the Office of Naval Research during the development of this book, including partial salary support for Xiaoming Wangs visit to Courant in the spring semester of 2001.

Contents

1

Barotropic geophysical flows and two-dimensional fluid flows: elementary introduction

1.1 Introduction

The atmosphere and the ocean are the two most important fluid systems of our planet. The bulk of the atmosphere is a thin layer of air 10 km thick that engulfs the earth, and the oceans cover about 70% of the surface of our planet. Both the atmosphere and the ocean are in states of constant motion where the main source of energy is supplied by the radiation of the sun. The large-scale motions of the atmosphere and the ocean constitute geophysical flows and the science that studies them is geophysical fluid dynamics. The motions of the atmosphere and the ocean become powerful mechanisms for the transport and redistribution of energy and matter. For example, the motion of cold and warm atmospheric fronts determine the local weather conditions; the warm waters of the Gulf Stream are responsible for the temperate climate in northern Europe; the winds and the currents transport the pollutants produced by industries. It is clear that the motions of the atmosphere and the ocean play a fundamental role in the dynamics of our planet and greatly affect the activities of mankind.

It is apparent that the dynamical processes involved in the description of geophysical flows in the atmosphere and the ocean are extremely complex. This is due to the large number of physical variables needed to describe the state of the system and the wide range of space and time scales involved in these processes. The physical variables may include the velocity, the pressure, the density, and, in addition, the humidity in the case of atmospheric motions or the salinity in the case of oceanic motions. The physical processes that determine the evolution of the geophysical flows are also numerous. They may include the Coriolis force due to the earth's rotation; the sun's radiation; the presence of topographical barriers, as represented by mountain ranges in the case of atmospheric flows and the ocean floor and the continental masses in the case of oceanic flows. There may be also dissipative energy mechanisms, for example due to eddy diffusivity or Ekman drag. The ranges of spatial and temporal scales involved in the description of geophysical flows is also very large. The space scales may vary from a few hundred meters to thousands of kilometers. Similarly, the time scales maybe as short as minutes and as long as days, months, or even years.

The above remarks make evident the need for simplifying assumptions regarding the relevant physical mechanisms involved in a given geophysical flow process, as well as the relevant range of space and time scales needed to describe the process. The treatises of Pedlosky (1987) and Gill (1982) are two excellent references to consult regarding the physical foundations of geophysical flows and different simplifying approximations utilized in the study of the various aspects of geophysical fluids. Here we concentrate on large-scale flows for the atmosphere or *mesoscale* flows in the oceans. The simplest set of equations that meaningfully describes the motion of geophysical flows under these circumstances is given by the:

Barotropic quasi-geostrophic equations

$$\begin{aligned}
&\frac{Dq}{Dt} = \mathcal{D}(\Delta)\psi + \mathcal{F}(\vec{x}, t) \\
&q = \omega + \beta y + h(x, y), \quad \text{where } \omega = \Delta\psi \\
&\vec{v} = \nabla^{\perp}\psi = \begin{pmatrix} -\dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial x} \end{pmatrix},
\end{aligned} \tag{1.1}$$

where $\frac{D}{Dt}$ stands for the advective (or material) derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$$

and Δ denotes the Laplacian operator

$$\Delta = \operatorname{div} \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In equation (1.1), q is the potential vorticity, $\vec{v}$ is the horizontal velocity field, ω, is the relative vorticity, and ψ is the stream function. The horizontal space variables are given by $\vec{x} = (x, y)$ and t denotes time. The term βy is called the beta-plane effect from the Coriolis force and its significance will be explained later. The term $h = h(x, y)$ represents the bottom floor topography. The term $\mathcal{D}(\Delta)\psi$ represents various possible dissipation mechanisms. Finally, the term $\mathcal{F}(\vec{x}, t)$ accounts for additional external forcing. The fluid density is set to 1.

Before continuing, we would like to explain briefly, in physical terms and without going into any technical details, the origin of the barotropic quasi-geostrophic equations. The barotropic rotational equations, also called rotating shallow water equations (Pedlosky, 1987), admit two different modes of propagation, slow and fast. The slow mode of propagation corresponds to the motion of the bulk of the fluid by advection. This is the slow motion we see in the weather patterns in the atmosphere, evolving on a time scale of days. The fast mode corresponds to gravity waves, which evolve on a short time scale of the order of several minutes, but do not contribute to the bulk motion of the fluid. The barotropic quasi-geostrophic equations are the result of "filtering out" the fast gravity waves from the rotating barotropic equations. There is also a formal analogy between barotropic quasi-geostrophic equations and incompressible flows; in the theory of compressible fluid flows the incompressible limit is obtained by "filtering out" the "fast" acoustic waves and retaining only the "slow" vortical modes associated to convection by the fluid (Majda, 1984). Indeed, it was this analogy that originally inspired Charney (1949) when he first formulated the quasi-geostrophic equations and thus opened the modern era of numerical weather prediction (Charney, 1949; Charney, Fjörtoft, and von Neumann, 1950).

The full derivation of the rotating barotropic equations and the corresponding barotropic quasi-geostrophic equations is lengthy and will take us too far from our main objective, which is the study of the quasi-geostrophic equations. For a thorough treatment of the barotropic rotational equations the reader is referred to Pedlosky (1987). Formal as well as rigorous derivations of the barotropic quasi-geostrophic equations from the rotating shallow water equations can be found in Majda (2003), Embid and Majda (1996).

Rather than deriving the quasi-geostrophic equations, we would like to explain the physical meaning and significance of the different terms appearing in equation (1.1). For barotropic quasi-geostrophic flows, the potential vorticity q is made of three different contributions. The first term $\omega = \Delta\psi = \text{curl}\,\vec{v}$ is the fluid vorticity and represents the local rate of rotation of the fluid. The second term βy is the beta-plane effect from the Coriolis force and its appearance will be explained later. The third term $h = h(x, y)$ represents the bottom topography, as given by the ocean floor or a mountain range.

The horizontal velocity field, $\vec{v}$, is determined by the orthogonal gradient of the stream function ψ, $\vec{v} = \nabla^{\perp}\psi$, where the orthogonal gradient of ψ is defined as

$$\nabla^{\perp}\psi = \begin{pmatrix} -\dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial x} \end{pmatrix}.$$

In particular, the velocity field $\vec{v}$ is incompressible because

$$\text{div}\,\vec{v} = \nabla\cdot\vec{v} = \nabla\cdot\nabla^{\perp}\psi = 0.$$

The reason ψ is called the stream function is because at any fixed instant in time the velocity field $\vec{v}$ is perpendicular to the gradient of ψ, i.e. $\vec{v}$ is tangent to the level curves of ψ. Therefore the level curves of ψ represent the streamlines of the fluid. In addition, there is another important interpretation of ψ. Physically ψ represents the (hydrostatic) pressure of the fluid. In this context, the equation $\vec{v} = \nabla^{\perp}\psi$ corresponds to the fact that the flow field is in *geostrophic balance*, and therefore the streamlines also happen to be the *isobars* of the flow. In particular, we conclude that for a steady solution of the quasi-geostrophic equations the fluid flows along the isobars. This is in marked contrast with the situation in non-rotating fluids, where typically the flow is from regions of high pressure to those of low pressure.

The importance of the potential vorticity q is in the fact that it completely determines the state of the flow. Indeed in the barotropic quasi-geostrophic equations, once we know the potential vorticity q, the second equation in equation (1.1) immediately yields the vorticity ω. Since $\omega = \Delta\psi$, we can determine the stream function ψ, and then introduce it into the third equation in equation (1.1), namely $\vec{v} = \nabla^{\perp}\psi$, to determine the advective velocity field.

Next we return to a brief discussion of the beta-plane effect (cf. Pedlosky, 1987). This effect is essentially the result of linearizing the Coriolis force when we consider the motion of the fluid in the tangent plane approximation. More specifically, although the earth is spherical, we assume that the spatial scale of motion is moderate enough so that the region occupied by the fluid can be approximated by a tangent plane (this is certainly the case for mesoscale flows, even for horizontal ranges of the order of 10^3 km). This is what is called the tangent plane approximation. The equations of motion in equation (1.1) are written in terms of horizontal Cartesian coordinates in the tangent plane. In this context, the spatial variable x corresponds to longitude (with positive direction towards the east) and the variable y to latitude (with positive direction towards the north).[1] In fact, throughout this book we often refer to flows pointing in the positive (negative) x-direction as eastward (westward). Since the tangent plane rotates with the earth it becomes a non-inertial frame, and the Coriolis force due to the earth's rotation becomes an important effect in geophysical flows. Moreover, because of the curvature of the earth, the contribution of the Coriolis force depends on the latitude at which the tangent plane is being considered; the Coriolis force will increase from zero at the equator to its maximum value at the poles. Since the tangent plane approximation assumes a moderate range in latitude and longitude, a Taylor expansion approximation of the Coriolis force is permissible; the linear term of this Taylor expansion yields the beta-plane effect βy considered in equation (1.1). For the actual details of the tangent plane approximation and the beta-plane effect, the reader is encouraged to consult Pedlosky (1987) or Gill (1982).

[1] For simplicity we will always assume that the tangent plane approximation is considered in the northern hemisphere

There are many choices of dissipation operator $\mathcal{D}(\Delta)$, ranging from Ekman drag to Newtonian viscosity or hyper-viscosity. We list some commonly used dissipation operators below for later convenience:

(i) Newtonian (eddy) viscosity

$$\mathcal{D}(\Delta)\psi = \nu\Delta^2\psi$$

This form of the diffusion is identical to the ordinary molecular friction in a Newtonian fluid. For geophysical flows, the value of the coefficient is often assumed to be many orders of magnitude larger than that for molecular viscosity, and represents, crudely, smaller-scale turbulence effects. This led to the name, eddy viscosity.

(ii) Ekman drag dissipation

$$\mathcal{D}(\Delta)\psi = -d\Delta\psi,$$

which is common to the large-scale pieces of the geophysical flow. This arises from boundary layer effects in rapidly rotating flows.

(iii) Hyper-viscosity dissipation

$$\mathcal{D}(\Delta)\psi = (-1)^j d_j \Delta^j \psi, \quad j = 3, 4, 5, \cdots$$

This form of the dissipation term is frequently utilized in the study and numerical simulation of geophysical flows, where its role is to introduce very little dissipation in the large scales of the flow but to strongly damp out the small scales. The validity of the use of such hyper-viscous mechanisms is still an open issue among geophysical fluid dynamicists.

(iv) Ekman drag dissipation + Hyper-viscosity

$$\mathcal{D}(\Delta)\psi = -d\Delta\psi + (-1)^j d_j \Delta^j \psi, \qquad d_j \geq 0, \quad d > 0, \quad j > 2.$$

This is a combination of the previous two dissipation mechanisms.

(v) Radiative damping

$$\mathcal{D}(\Delta)\psi = d\psi$$

This represents a crude model for radiative damping when models with stratification are involved. Radiative damping is an unusual dissipation operator since it damps the large scales more strongly than the small scales in contrast to the standard diffusion operators in (i) and (iii) above.

(vi) General dissipation operator

$$\mathcal{D}(\Delta)\psi = \sum_{j=0}^{l} (-1)^j d_j \Delta^j \psi,$$

which encompasses all other forms of dissipation mechanisms previously discussed.

For simplicity we will consider periodic boundary conditions for the flow in both the x and y variables, say with period 2π in both variables

$$\begin{aligned} \vec{v}(x+2\pi, y, t) &= \vec{v}(x, y, t), \\ \vec{v}(x, y+2\pi, t) &= \vec{v}(x, y, t), \end{aligned} \tag{1.2}$$

or in terms of the stream function ψ

$$\psi(x+2\pi, y, t) = \psi(x, y+2\pi, t) = \psi(x, y, t). \tag{1.3}$$

We may also impose the zero average condition

$$\int \psi(x, y, t)\, dxdy = 0, \tag{1.4}$$

since the stream function is always determined up to a constant, and we can choose the constant here so that the average is zero. The assumption of periodicity in both variables is not unreasonable (except near drastic topographical barriers, such as continents). It allows us to use Fourier series and separation of variables as a main mathematical tool (see page 10 for a Fourier series tool kit). Physically, periodicity allows us to avoid other issues such as the appearance of boundary layers or the generation of vorticity at the boundary. However, occasionally we will consider other boundary conditions besides the periodic one. In particular, we will study flows in channel domains or in a rectangular basin which can be treated through minor modification of periodic flows with special geometry.

It is worthwhile to point out that, in the special case where there are no beta-plane effects or bottom floor topography, i.e. $\beta = 0$, $h = 0$, then the potential vorticity q reduces to the vorticity ω, $q = \omega$, and if we assume Newtonian dissipation, then the barotropic quasi-geostrophic equations reduce to the classical Navier–Stokes equations for a two-dimensional flow, written in the vorticity-stream form (Majda and Bertozzi, 2001; Chorin and Marsden, 1993)

Two-dimensional classical fluid flow equations

$$\frac{D\omega}{Dt} = \nu\Delta\omega + \mathcal{F}(\vec{x}, t), \quad \omega = \Delta\psi, \quad \vec{v} = \nabla^{\perp}\psi \tag{1.5}$$

and in the case without dissipation we have the classical Euler equations with forcing

$$\frac{D\omega}{Dt} = \mathcal{F}(\vec{x}, t), \quad \omega = \Delta\psi, \quad \vec{v} = \nabla^{\perp}\psi. \tag{1.6}$$

One of our objectives of this book is to compare and contrast the barotropic quasi-geostrophic equations and the Navier–Stokes equations to better understand the role of the beta-plane effect and the topography on the behavior of geophysical flows.