Graduate Texts in Mathematics

Hershel M.Farkas
Irwin Kra

Riemann Surfaces

Second Edition

黎曼曲面 第2版

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To
Eleanor
Sara

Preface to the Second Edition

It is gratifying to learn that there is new life in an old field that has been at the center of one's existence for over a quarter of a century. It is particularly pleasing that the subject of Riemann surfaces has attracted the attention of a new generation of mathematicians from (newly) adjacent fields (for example, those interested in hyperbolic manifolds and iterations of rational maps) and young physicists who have been convinced (certainly not by mathematicians) that compact Riemann surfaces may play an important role in their (string) universe. We hope that non-mathematicians as well as mathematicians (working in nearby areas to the central topic of this book) will also learn part of this subject for the sheer beauty and elegance of the material (work of Weierstrass, Jacobi, Riemann, Hilbert, Weyl) and as healthy exposure to the way (some) mathematicians write about mathematics.

We had intended a more comprehensive revision, including a fuller treatment of moduli problems and theta functions. Pressure of other commitments would have substantially delayed (by years) the appearance of the book we wanted to produce. We have chosen instead to make a few modest additions and to correct a number of errors. We are grateful to the readers who pointed out some of our mistakes in the first edition; the responsibility for the remaining mistakes carried over from the first edition and for any new ones introduced into the second edition remains with the authors.

June 1991
Jerusalem
and
Stony Brook

H.M. Farkas
and
I. Kra

Preface to the First Edition

The present volume is the culmination of ten years' work separately and jointly. The idea of writing this book began with a set of notes for a course given by one of the authors in 1970–1971 at the Hebrew University. The notes were refined several times and used as the basic content of courses given subsequently by each of the authors at the State University of New York at Stony Brook and the Hebrew University.

In this book we present the theory of Riemann surfaces and its many different facets. We begin from the most elementary aspects and try to bring the reader up to the frontier of present-day research. We treat both open and closed surfaces in this book, but our main emphasis is on the compact case. In fact, Chapters III, V, VI, and VII deal exclusively with compact surfaces. Chapters I and II are preparatory, and Chapter IV deals with uniformization.

All works on Riemann surfaces go back to the fundamental results of Riemann, Jacobi, Abel, Weierstrass, etc. Our book is no exception. In addition to our debt to these mathematicians of a previous era, the present work has been influenced by many contemporary mathematicians.

At the outset we record our indebtedness to our teachers Lipman Bers and Harry Ernest Rauch, who taught us a great deal of what we know about this subject, and who along with Lars V. Ahlfors are responsible for the modern rebirth of the theory of Riemann surfaces. Second, we record our gratitude to our colleagues whose theorems we have freely written down without attribution. In particular, some of the material in Chapter III is the work of Henrik H. Martens, and some of the material in Chapters V and VI ultimately goes back to Robert D. M. Accola and Joseph Lewittes.

We thank several colleagues who have read and criticized earlier versions of the manuscript and made many helpful suggestions: Bernard Maskit,

Henry Laufer, Uri Srebro, Albert Marden, and Frederick P. Gardiner. The errors in the final version are, however, due only to the authors. We also thank the secretaries who typed the various versions: Carole Alberghine and Estella Shivers.

August 1979

H.M. FARKAS I. KRA

Commonly Used Symbols

$\mathbb{Z}$	integers
$\mathbb{Q}$	rationals
$\mathbb{R}$	real numbers
$\mathbb{R}^n$	n-dimensional real Euclidean spaces
$\mathbb{C}^n$	n-dimensional complex Euclidean spaces
Re	real part
Im	imaginary part
$\lvert\cdot\rvert$	absolute value
C^∞	infinitely differentiable (function or differential)
$\mathscr{H}^q(M)$	linear space of holomorphic q-differentials on M
$\mathscr{K}(M)$	field of meromorphic functions on M
deg	degree of divisor or map
$L(D)$	linear space of the divisor D
$r(D)$	$\dim L(D) =$ dimension of D
$\Omega(D)$	space of meromorphic abelian differentials of the divisor D
$i(D)$	$\dim \Omega(D) =$ index of specialty of D
[ ]	greatest integer in
$c(D)$	Clifford index of D
$\mathrm{ord}_P f$	order of f at P
$\pi_1(M)$	fundamental group of M
$H_1(M)$	first (integral) homology group of M

$J(M)$	Jacobian variety of M
Π	period matrix of M
M_n	integral divisors of degree n on M
M_n^r	$\{D \in M_n; r(D^{-1}) \geq r + 1\}$
W_n	image of M_n in $J(M)$
W_n^r	image of M_n^r in $J(M)$
Z	canonical divisor
K	vector of Riemann constants (usually)
${}^t x$	transpose of the matrix x (vectors are usually written as columns; thus for $x \in \mathbb{R}^n$, ${}^t x$ is a row vector)

Contents

CHAPTER 0
An Overview

The theory of Riemann surfaces lies in the intersection of many important areas of mathematics. Aside from being an important field of study in its own right, it has long been a source of inspiration, intuition, and examples for many branches of mathematics. These include complex manifolds, Lie groups, algebraic number theory, harmonic analysis, abelian varieties, algebraic topology.

The development of the theory of Riemann surfaces consists of at least three parts: a topological part, an algebraic part, and an analytic part. In this chapter, we shall try to outline how Riemann surfaces appear quite naturally in different guises, list some of the most important problems to be treated in this book, and discuss the solutions.

As the title indicates, this chapter is a survey of results. Many of the statements are major theorems. We have indicated at the end of most paragraphs a reference to subsequent chapters where the theorem in question is proven or a fuller discussion of the given topic may be found. For some easily verifiable claims a (kind of) proof has been supplied. This chapter has been written for the reader who wishes to get an idea of the scope of the book before entering into details. It can be skipped, since it is independent of the formal development of the material. This chapter is intended primarily for the mathematician who knows other areas of mathematics and is interested in finding out what the theory of Riemann surfaces contains. The graduate student who is familiar only with first year courses in algebra, analysis (real and complex), and algebraic topology should probably skip most of this chapter and periodically return to it.

We, of course, begin with a definition: A *Riemann surface* is a complex 1-dimensional connected (analytic) manifold.

0.1. Topological Aspects, Uniformization, and Fuchsian Groups

Given a connected topological manifold M (which in our case is a Riemann surface), one can always construct a new manifold $\tilde{M}$ known as the universal covering manifold of M. The manifold $\tilde{M}$ has the following properties:

1. There is a surjective local homeomorphism $\pi:\tilde{M} \to M$.
2. The manifold $\tilde{M}$ is simply connected; that is, the fundamental group of $\tilde{M}$ is trivial ($\pi_1(\tilde{M}) = \{1\}$).
3. Every closed curve which is not homotopically trivial on M lifts to an open curve on $\tilde{M}$, and the curve on $\tilde{M}$ is uniquely determined by the curve on M and the point lying over its initial point.

In fact one can say a lot more. If M^* is any covering manifold of M, then $\pi_1(M^*)$ is isomorphic to a subgroup of $\pi_1(M)$. The covering manifolds of M are in bijective correspondence with conjugacy classes of subgroups of $\pi_1(M)$. In this setting, $\tilde{M}$ corresponds to the trivial subgroup of $\pi_1(M)$. Furthermore, in the case that the subgroup N of $\pi_1(M)$ is normal, there is a group $G \cong \pi_1(M)/N$ of fixed point free automorphisms of M^* such that $M^*/G \cong M$. Once again in the case of the universal covering manifold $\tilde{M}$, $G \cong \pi_1(M)$. (I.2.4; IV.5.6)

If we now make the assumption that M is a Riemann surface, then it is not hard to introduce a Riemann surface structure on any M^* in such a way that the map $\pi:M^* \to M$ becomes a holomorphic mapping between Riemann surfaces and G becomes a group of holomorphic self-mappings of M^* such that $M^*/G \cong M$. (IV.5.5–IV.5.7)

It is at this point that some analysis has to intervene. It is necessary to find all the simply connected Riemann surfaces. The result is both beautiful and elegant. There are exactly three conformally (= complex analytically) distinct simply connected Riemann surfaces. One of these is compact, it is conformally equivalent to the sphere $\mathbb{C} \cup \{\infty\}$. The non-compact simply connected Riemann surfaces are conformally equivalent to either the upper half plane U or the entire plane $\mathbb{C}$. (IV.4)

It thus follows from what we have said before that studying Riemann surfaces is essentially the same as studying fixed point free discontinuous groups of holomorphic self mappings of D, where D is either $\mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, or U. (IV.5.5)

The simplest case occurs when $D = \mathbb{C} \cup \{\infty\}$. Since every non-trivial holomorphic self map of $\mathbb{C} \cup \{\infty\}$ has at least one fixed point, only the sphere covers the sphere. (IV.6.3)

The holomorphic fixed point free self maps of $\mathbb{C}$ are of the form $z \mapsto z + b$, with $b \in \mathbb{C}$. An analysis of the various possibilities shows that a discontinuous subgroup of this group is either trivial or cyclic on one (free) generator or a free abelian group with two generators. The first case