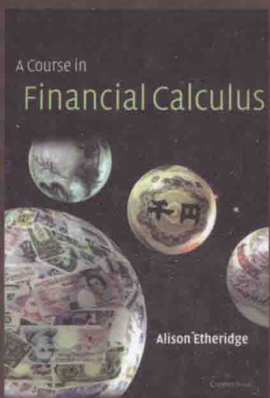
TURING 图灵原版数学·统计学系列

A Course in Financial Calculus

金融数学教程

（英文版）

[英] Alison Etheridge 著



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内 容 提 要

金融为现代数学技术成功地应用于实际问题提供了一个十分生动的例子：金融衍生品定价。本书可作为金融数学入门教材，含有大量的习题和例子，面向有一定数学基础的读者。本书首先基于离散时间框架介绍了一些基本概念，如二叉树、鞅、布朗运动、随机积分及 Black-Scholes 期权定价公式，然后介绍了一些复杂的金融模型和金融产品，最后一章则介绍了金融方面更为高级的话题，如带跳的股票价格模型和随机波动率等。

本书作为金融数学的基础教材，适用于相关专业的本科生和研究生课程，也可供相关领域专业人士参考。

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Preface

Financial mathematics provides a striking example of successful collaboration between academia and industry. Advanced mathematical techniques, developed in both universities and banks, have transformed the derivatives business into a multi-trillion-dollar market. This has led to demand for highly trained students and with that demand comes a need for textbooks.

This volume provides a first course in financial mathematics. The influence of *Financial Calculus* by Martin Baxter and Andrew Rennie will be obvious. I am extremely grateful to Martin and Andrew for their guidance and for allowing me to use some of the material from their book.

The structure of the text largely follows *Financial Calculus*, but the mathematics, especially the discussion of stochastic calculus, has been expanded to a level appropriate to a university mathematics course and the text is supplemented by a large number of exercises. In order to keep the course to a reasonable length, some sacrifices have been made. Most notable is that there was not space to discuss interest rate models, although many of the most popular ones do appear as examples in the exercises. As partial compensation, the necessary mathematical background for a rigorous study of interest rate models is included in Chapter 7, where we briefly discuss some of the topics that one might hope to include in a *second* course in financial mathematics. The exercises should be regarded as an integral part of the course. Solutions to these are available to *bona fide* teachers from solutions@cambridge.org.

The emphasis is on stochastic techniques, but not to the exclusion of all other approaches. In common with practically every other book in the area, we use binomial trees to introduce the ideas of arbitrage pricing. Following *Financial Calculus*, we also present discrete versions of key definitions and results on martingales and stochastic calculus in this simple framework, where the important ideas are not obscured by analytic technicalities. This paves the way for the more technical results of later chapters. The connection with the partial differential equation approach to arbitrage pricing is made through both delta-hedging arguments and the Feynman–Kac Stochastic Representation Theorem. Whatever approach one adopts, the key point that we wish to emphasise is that since the theory rests on the assumption of absence of arbitrage, hedging is vital. Our pricing formulae only make sense if there is a 'replicating portfolio'.

An early version of this course was originally delivered to final year undergraduate and first year graduate mathematics students in Oxford in 1997/8. Although we assumed some familiarity with probability theory, this was not regarded as a prerequisite and students on those courses had little difficulty picking up the necessary concepts as we met them. Some suggestions for suitable background reading are made in the bibliography. Since a first course can do little more than scratch the surface of the subject, we also make suggestions for supplementary and more advanced reading from the bewildering array of available books.

This project was supported by an EPSRC Advanced Fellowship. It is a pleasure and a privilege to work in Magdalen College and my thanks go to the President, Fellows, staff and students for making it such an exceptional environment. Many people have made helpful suggestions or read early drafts of this volume. I should especially like to thank Ben Hambly, Alex Jackson and Saurav Sen. Thanks also to David Tranah at CUP who played a vital rôle in shaping the project. His input has been invaluable. Most of all, I should like to thank Lionel Mason for his constant support and encouragement.

Alison Etheridge, June 2001

Contents

1 Single period models

Summary

In this chapter we introduce some basic definitions from finance and investigate the problem of pricing financial instruments in the context of a very crude model. We suppose the market to be observed at just two times: zero, when we enter into a financial contract; and T, the time at which the contract expires. We further suppose that the market can only be in one of a finite number of states at time T. Although simplistic, this model reveals the importance of the central paradigm of modern finance: the idea of a perfect hedge. It is also adequate for a preliminary discussion of the notion of 'complete market' and its importance if we are to find a 'fair' price for our financial contract.

The proofs in §1.5 can safely be omitted, although we shall from time to time refer back to the statements of the results.

1.1 Some definitions from finance

Financial market instruments can be divided into two types. There are the *underlying* stocks – shares, bonds, commodities, foreign currencies; and their *derivatives*, claims that promise some payment or delivery in the future contingent on an underlying stock's behaviour. Derivatives can reduce risk – by enabling a player to fix a price for a future transaction now – or they can magnify it. A costless contract agreeing to pay off the difference between a stock and some agreed future price lets both sides ride the risk inherent in owning a stock, without needing the capital to buy it outright.

The connection between the two types of instrument is sufficiently complex and uncertain that both trade fiercely in the same market. The apparently random nature of the underlying stocks filters through to the derivatives – they appear random too.

Derivatives

Our central purpose is to determine how much one should be willing to pay for a derivative security. But first we need to learn a little more of the language of finance.

Definition 1.1.1 *A* forward contract *is an agreement to buy (or sell) an asset on a specified future date, T, for a specified price, K. The buyer is said to hold the* long *position, the seller the* short *position.*

Forwards are not generally traded on exchanges. It costs nothing to enter into a forward contract. The 'pricing problem' for a forward is to determine what value of K should be written into the contract. A *futures contract* is the same as a forward except that futures *are* normally traded on exchanges and the exchange specifies certain standard features of the contract and a particular form of settlement.

Forwards provide the simplest examples of derivative securities and the mathematics of the corresponding pricing problem will also be simple. A much richer theory surrounds the pricing of *options*. An option gives the holder the *right*, but not the *obligation*, to do something. Options come in many different guises. Black and Scholes gained fame for pricing a European call option.

Definition 1.1.2 *A* European call option *gives the holder the right, but not the obligation, to buy an asset at a specified time, T, for a specified price, K.*

A European put option *gives the holder the right to* sell *an asset for a specified price, K, at time T.*

In general *call* refers to buying and *put* to selling. The term *European* is reserved for options whose value to the holder at the time, T, when the contract expires depends on the state of the market only at time T. There are other options, for example American options or Asian options, whose payoff is contingent on the behaviour of the underlying over the whole time interval $[0, T]$, but the technology of this chapter will only allow meaningful discussion of European options.

Definition 1.1.3 *The time, T, at which the derivative contract expires is called the* exercise date *or the* maturity. *The price K is called the* strike price.

The pricing problem

So what is the pricing problem for a European call option? Suppose that a company has to deal habitually in an intrinsically risky asset such as oil. They may for example know that in three months time they will need a thousand barrels of crude oil. Oil prices can fluctuate wildly, but by purchasing European call options, with strike K say, the company knows the *maximum* amount of money that it will need (in three months time) in order to buy a thousand barrels. One can think of the option as insurance against increasing oil prices. The pricing problem is now to determine, for given T and K, how much the company should be willing to pay for such insurance.

For this example there is an extra complication: it costs money to store oil. To simplify our task we are first going to price derivatives based on assets that can be held without additional cost, typically company shares. Equally we suppose that there is no additional benefit to holding the shares, that is no dividends are paid.

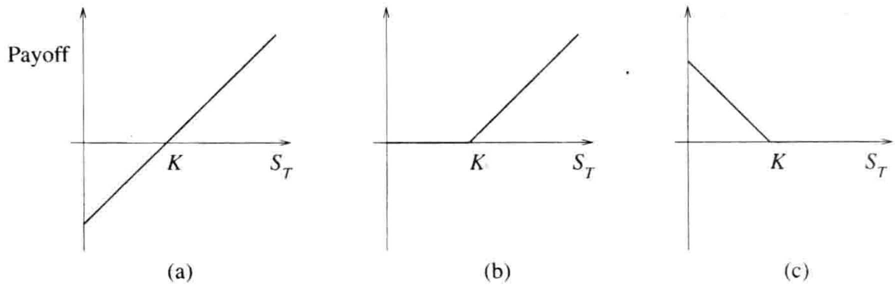

Figure 1.1 Payoff at maturity of (a) a forward purchase, (b) a European call and (c) a European put with strike K as a function of S_T.

> **Assumption** Unless otherwise stated, the underlying asset can be held without additional cost or benefit.

This assumption will be relaxed in Chapter 5.

Suppose then that our company enters into a contract that gives them the right, but not the obligation, to buy one unit of stock for price K in three months time. How much should they pay for this contract?

Payoffs

As a first step, we need to know what the contract will be worth at the expiry date. If at the time when the option expires (three months hence) the actual price of the underlying stock is S_T and $S_T > K$ then the option will be exercised. The option is then said to be *in the money*: an asset worth S_T can be purchased for just K. The value to the company of the option is then $(S_T - K)$. If, on the other hand, $S_T < K$, then it will be cheaper to buy the underlying stock on the open market and so the option will not be exercised. (It is this freedom *not* to exercise that distinguishes options from futures.) The option is then worthless and is said to be *out of the money*. (If $S_T = K$ the option is said to be *at the money*.) The *payoff* of the option at time T is thus

$$(S_T - K)_+ \triangleq \max\{(S_T - K), 0\}.$$

Figure 1.1 shows the payoff at maturity of three derivative securities: a forward purchase, a European call and a European put, each as a function of stock price at maturity. Before embarking on the valuation *at time zero* of derivative contracts, we allow ourselves a short aside.

Packages

We have presented the European call option as a means of reducing risk. Of course it can also be used by a speculator as a bet on an increase in the stock price. In fact by holding *packages*, that is combinations of the 'vanilla' options that we have described so far, we can take rather complicated bets. We present just one example; more can be found in Exercise 1.

Example 1.1.4 (A straddle) *Suppose that a speculator is expecting a large move in a stock price, but does not know in which direction that move will be. Then a possible combination is a* straddle. *This involves holding a European call and a European put with the same strike price and maturity.*

Explanation: The payoff of this straddle is $(S_T - K)_+$ (from the call) plus $(K - S_T)_+$ (from the put), that is, $|S_T - K|$. Although the payoff of this combination is always positive, if, at the expiry time, the stock price is too close to the strike price then the payoff will not be sufficient to offset the cost of purchasing the options and the investor makes a loss. On the other hand, large movements in price can lead to substantial profits. □

1.2 Pricing a forward

In order to solve our pricing problems, we are going to have to make some assumptions about the way in which markets operate. To formulate these we begin by discussing forward contracts in more detail.

Recall that a forward contract is an agreement to buy (or sell) an asset on a specified future date for a specified price. Suppose then that I agree to buy an asset for price K at time T. The payoff at time T is just $S_T - K$, where S_T is the actual asset price at time T. The payoff could be positive or it could be negative and, since the cost of entering into a forward contract is zero, this is also my total gain (or loss) from the contract. Our problem is to determine the fair value of K.

Expectation pricing

At the time when the contract is written, we don't know S_T, we can only guess at it, or, more formally, assign a probability distribution to it. A widely used model (which underlies the Black–Scholes analysis of Chapter 5) is that stock prices are *lognormally distributed*. That is, there are constants ν and σ such that the *logarithm* of S_T/S_0 (the stock price at time T divided by that at time zero, usually called the *return*) is normally distributed with mean ν and variance σ^2. In symbols:

$$\begin{aligned}\mathbb{P}\left[\frac{S_T}{S_0} \in [a, b]\right] &= \mathbb{P}\left[\log\left(\frac{S_T}{S_0}\right) \in [\log a, \log b]\right] \\ &= \int_{\log a}^{\log b} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) dx.\end{aligned}$$

Notice that stock prices, and therefore a and b, should be positive, so that the integral on the right hand side is well defined.

Our first guess might be that $\mathbb{E}[S_T]$ should represent a fair price to write into our contract. However, it would be a rare coincidence for this to be the market price. In fact we'll show that the cost of borrowing is the key to our pricing problem.

The risk-free rate

We need a model for the *time value of money*: a dollar now is worth more than a dollar promised at some later time. We assume a market for these future promises (the *bond* market) in which prices are derivable from some interest rate. Specifically:

> **Time value of money** We assume that for any time T less than some horizon τ the value now of a dollar promised at T is e^{-rT} for some constant $r > 0$. The rate r is then the *continuously compounded* interest rate for this period.

Such a market, derived from say US Government bonds, carries no risk of default – the promise of a future dollar will always be honoured. To emphasise this we will often refer to r as the *risk-free interest rate*. In this model, by buying or selling cash bonds, investors can borrow money for the same risk-free rate of interest as they can lend money.

Interest rate markets are not this simple in practice, but that is an issue that we shall defer.

Arbitrage pricing

We now show that it is the *risk-free interest rate*, or equivalently the price of a cash bond, and not our lognormal model that forces the choice of the strike price, K, upon us in our forward contract.

Interest rates will be different for different currencies and so, for definiteness, suppose that we are operating in the dollar market, where the (risk-free) interest rate is r.

- Suppose first that $K > S_0e^{rT}$. The seller, obliged to deliver a unit of stock for $\$K$ at time T, adopts the following strategy: she borrows $\$S_0$ at time zero (i.e. sells bonds to the value $\$S_0$) and buys one unit of stock. At time T, she must repay $\$S_0e^{rT}$, but she has the stock to sell for $\$K$, leaving her a *certain* profit of $\$(K - S_0e^{rT})$.
- If $K < S_0e^{rT}$, then the buyer reverses the strategy. She sells a unit of stock at time zero for $\$S_0$ and buys cash bonds. At time T, the bonds deliver $\$S_0e^{rT}$ of which she uses $\$K$ to buy back a unit of stock leaving her with a *certain* profit of $\$(S_0e^{rT} - K)$.

Unless $K = S_0e^{rT}$, one party is guaranteed to make a profit.

Definition 1.2.1 *An opportunity to lock into a risk-free profit is called an* arbitrage *opportunity.*

The starting point in establishing a model in modern finance theory is to specify that there is no arbitrage. (In fact there are people who make their living entirely from exploiting arbitrage opportunities, but such opportunities do not exist for a significant length of time before market prices move to eliminate them.) We have proved the following lemma.

Lemma 1.2.2 *In the absence of arbitrage, the strike price in a forward contract with expiry date T on a stock whose value at time zero is S_0 is $K = S_0e^{rT}$, where r is the risk-free rate of interest.*

The price S_0e^{rT} is sometimes called the *arbitrage price*. It is also known as the *forward price* of the stock.

Remark: In our proof of Lemma 1.2.2, the buyer sold stock that she may not own. This is known as *short selling*. This can, and does, happen: investors can 'borrow' stock as well as money. □

Of course forwards are a very special sort of derivative. The argument above won't tell us how to value an option, but the strategy of seeking a price that does not provide either party with a risk-free profit will be fundamental in what follows.

Let us recap what we have done. In order to price the forward, we constructed a portfolio, comprising one unit of underlying stock and $-S_0$ cash bonds, whose value at the maturity time T is *exactly* that of the forward contract itself. Such a portfolio is said to be a *perfect hedge* or *replicating portfolio*. This idea is the central paradigm of modern mathematical finance and will recur again and again in what follows. Ironically we shall use expectation repeatedly, but as a tool in the construction of a perfect hedge.

1.3 The one-step binary model

We are now going to turn to establishing the fair price for European call options, but in order to do so we first move to a *simpler* model for the movement of market prices. Once again we suppose that the market is observed at just two times, that at which the contract is struck and the expiry date of the contract. Now, however, we shall suppose that there are just two possible values for the stock price at time T. We begin with a simple example.

Pricing a European call

Example 1.3.1 *Suppose that the current price in Japanese Yen of a certain stock is* ¥2500. *A European call option, maturing in six months time, has strike price* ¥3000. *An investor believes that with probability one half the stock price in six months time will be* ¥4000 *and with probability one half it will be* ¥2000. *He therefore calculates the expected value of the option (when it expires) to be* ¥500. *The riskless borrowing rate in Japan is currently zero and so he agrees to pay* ¥500 *for the option. Is this a fair price?*

Solution: In the light of the previous section, the reader will probably have guessed that the answer to this question is no. Once again, we show that one party to this contract can make a risk-free profit. In this case it is the seller of the contract. Here is just one of the many possible strategies that she could adopt.

Strategy: At time zero, sell the option, borrow ¥2000 and buy a unit of stock.

- Suppose first that at expiry the price of the stock is ¥4000; then the contract will be exercised and so she must sell her stock for ¥3000. She then holds ¥$(-2000+3000)$. That is ¥1000.
- If, on the other hand, at expiry the price of the stock is ¥2000, then the option will not be exercised and so she sells her stock on the open market for just ¥2000. Her

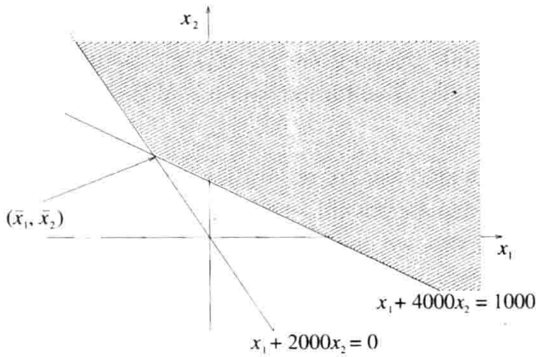

Figure 1.2 The seller of the contract in Example 1.3.1 is guaranteed a risk-free profit if she can buy any portfolio in the shaded region.

net cash holding is then ¥$(-2000 + 2000)$. That is, she exactly breaks even.

Either way, our seller has a positive chance of making a profit with *no risk* of making a loss. The price of the option is too high.

So what is the right price for the option?

Let's think of things from the point of view of the seller. Writing S_T for the price of the stock when the contract expires, she knows that at time T she needs ¥$(S_T - 3000)_+$ in order to meet the claim against her. The idea is to calculate how much money she needs at time zero, to be held in a combination of stocks and cash, to guarantee this.

Suppose then that she uses the money that she receives for the option to buy a portfolio comprising x_1 Yen and x_2 stocks. If the price of the stock is ¥4000 at expiry, then the time T value of the portfolio is $x_1 e^{rT} + 4000x_2$. The seller of the option requires this to be at least ¥1000. That is, since interest rates are zero,

$$x_1 + 4000x_2 \geq 1000.$$

If the price is ¥2000 she just requires the value of the portfolio to be non-negative,

$$x_1 + 2000x_2 \geq 0.$$

A profit is guaranteed (without risk) for the seller if (x_1, x_2) lies in the interior of the shaded region in Figure 1.2. On the boundary of the region, there is a positive probability of profit and no probability of loss at all points other than the intersection of the two lines. The portfolio represented by the point $(\overline{x}_1, \overline{x}_2)$ will provide *exactly* the wealth required to meet the claim against her at time T.

Solving the simultaneous equations gives that the seller can exactly meet the claim if $\overline{x}_1 = -1000$ and $\overline{x}_2 = 1/2$. The cost of building this portfolio at time zero is ¥$(-1000 + 2500/2)$, that is ¥250. For any price higher than ¥250, the seller can make a risk-free profit.

If the option price is *less* than ¥250, then the *buyer* can make a risk-free profit by 'borrowing' the portfolio $(\overline{x}_1, \overline{x}_2)$ and buying the option. In the absence of arbitrage then, the fair price for the option is ¥250. □

Notice that just as for our forward contract, we did not use the probabilities that we assigned to the possible market movements to arrive at the fair price. We just needed the fact that we could *replicate* the claim by this simple portfolio. The seller can *hedge* the *contingent claim* $¥(S_T - 3000)_+$ using the *portfolio* consisting of $¥x_1$ and x_2 units of stock.

Pricing formula for European options

One can use exactly the same argument to prove the following result.

Lemma 1.3.2 *Suppose that the risk-free dollar interest rate (to a time horizon $\tau > T$) is r. Denote the time zero (dollar) value of a certain asset by S_0. Suppose that the motion of stock prices is such that the value of the asset at time T will be either $S_0 u$ or $S_0 d$. Assume further that*

$$d < e^{rT} < u.$$

At time zero, the market price of a European option with payoff $C(S_T)$ at the maturity T is

$$\left(\frac{1 - de^{-rT}}{u - d}\right) C\left(S_0 u\right) + \left(\frac{ue^{-rT} - 1}{u - d}\right) C\left(S_0 d\right).$$

Moreover, the seller of the option can construct a portfolio whose value at time T is exactly $(S_T - K)_+$ by using the money received for the option to buy

$$\phi \triangleq \frac{C\left(S_0 u\right) - C\left(S_0 d\right)}{S_0 u - S_0 d} \tag{1.1}$$

units of stock at time zero and holding the remainder in bonds.

The proof is Exercise 4(a).

1.4 A ternary model

There were several things about the binary model that were very special. In particular we assumed that we knew that the asset price would be one of just two specified values at time T. What if we allow *three* values?

We can try to repeat the analysis of §1.3. Again the seller would like to replicate the claim at time T by a portfolio consisting of $¥x_1$ and x_2 stocks. This time there will be three scenarios to consider, corresponding to the three possible values of S_T. If interest rates are zero, this gives rise to the three inequalities

$$x_1 + S_T^i x_2 \geq (S_T^i - 3000)_+, \qquad i = 1, 2, 3,$$

where S_T^i are the possible values of S_T. The picture is now something like that in Figure 1.3.

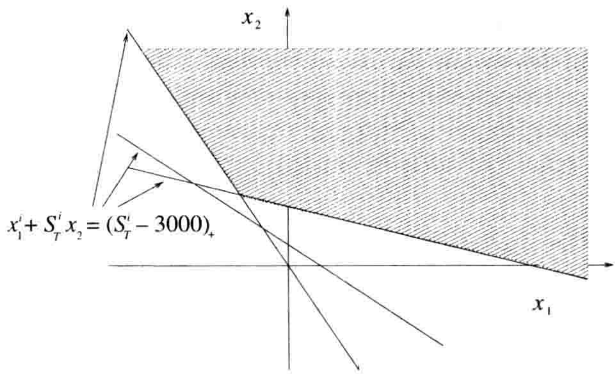

Figure 1.3 If the stock price takes three possible values at time T, then at any point where the seller of the option has no risk of making a loss, she has a *strictly positive* chance of making a profit.

In order to be *guaranteed* to meet the claim at time T, the seller requires (x_1, x_2) to lie in the shaded region, but at any point in that region, she has a strictly positive probability of making a profit and zero probability of making a loss. Any portfolio from outside the shaded region carries a risk of a loss. There is no portfolio that *exactly* replicates the claim and there is no unique 'fair' price for the option.

Our market is not *complete*. That is, there are contingent claims that cannot be perfectly hedged.

Bigger models

Of course we are tying our hands in our efforts to hedge a claim. First, we are only allowing ourselves portfolios consisting of the underlying stock and cash bonds. Real markets are bigger than this. If we allow ourselves to trade in a third 'independent' asset, then our analysis leads to three non-parallel planes in $\mathbb{R}^3$. These *will* intersect in a single point representing a portfolio that exactly replicates the claim. This then raises a question: when is there arbitrage in larger market models? We shall answer this question for a single period model in the next section. The second constraint that we have placed upon ourselves is that we are not allowed to adjust our portfolio between the time of the selling of the contract and its maturity. In fact, as we see in Chapter 2, if we consider the market to be observable at intermediate times between zero and T, and allow our seller to rebalance her portfolio at such times (without changing its value), then we *can* allow any number of possible values for the stock price at time T and yet still replicate each claim at time T by a portfolio consisting of just the underlying and cash bonds.

1.5 A characterisation of no arbitrage

In our binary setting it was easy to find the right price for an option simply by solving a pair of simultaneous equations. However, the binary model is very special and, after our experience with the ternary model, alarm bells may be ringing. The binary model describes the evolution of just one stock (and one bond). One solution to our