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实分析和概率论

(英文版·第2版)

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Real Analysis and Probability

R. M. DUDLEY

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麻省理工学院

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Preface to the Cambridge Edition

This is a text at the beginning graduate level. Some study of intermediate analysis in Euclidean spaces will provide helpful background, but in this edition such background is not a formal prerequisite. Efforts to make the book more self-contained include inserting material on the real number system into Chapter 1, adding a treatment of the Stone-Weierstrass theorem, and generally eliminating references for proofs to other books except at very few points, such as some complex variable theory in Appendix B.

Chapters 1 through 5 provide a one-semester course in real analysis. Following that, a one-semester course on probability can be based on Chapters 8 through 10 and parts of 11 and 12. Starred paragraphs and sections, such as those found in Chapter 6 and most of Chapter 7, are called on rarely, if at all, later in the book. They can be skipped, at least on first reading, or until needed.

Relatively few proofs of less vital facts have been left to the reader. I would be very glad to know of any substantial unintentional gaps or errors. Although I have worked and checked all the problems and hints, experience suggests that mistakes in problems, and hints that may mislead, are less obvious than errors in the text. So take hints with a grain of salt and perhaps make a first try at the problems without using the hints.

I looked for the best and shortest available proofs for the theorems. Short proofs that have appeared in journal articles, but in few if any other textbooks, are given for the completion of metric spaces, the strong law of large numbers, the ergodic theorem, the martingale convergence theorem, the subadditive ergodic theorem, and the Hartman-Wintner law of the iterated logarithm.

Around 1950, when Halmos' classic *Measure Theory* appeared, the more advanced parts of the subject headed toward measures on locally compact spaces, as in, for example, §7.3 of this book. Since then, much of the research in probability theory has moved more in the direction of metric spaces. Chapter 11 gives some facts connecting metrics and probabilities which follow the newer trend. Appendix E indicates what can go wrong with measures on (locally) compact nonmetric spaces. These parts of the book may well not be reached in a typical one-year course but provide some distinctive material for present and future researchers.

Problems appear at the end of each section, generally increasing in difficulty as they go along. I have supplied hints to the solution of many of the problems. There are a lot of new or, I hope, improved hints in this edition.

I have also tried to trace back the history of the theorems to give credit where it is due. Historical notes and references, sometimes rather extensive, are given at the end of each chapter. Many of the notes have been augmented in this edition and some have been corrected. I don't claim, however, to give the last word on any part of the history.

The book evolved from courses given at M.I.T. since 1967 and in Aarhus, Denmark, in 1976. For valuable comments I am glad to thank Ken Alexander, Deborah Allinger, Laura Clemens, Ken Davidson, Don Davis, Persi Diaconis, Arnout Eikeboom, Sy Friedman, David Gillman, José Gonzalez, E. Griffor, Leonid Grinblat, Dominique Haughton, J. Hoffmann-Jørgensen, Arthur Mattuck, Jim Munkres, R. Proctor, Nick Reingold, Rae Shortt, Dorothy Maharam Stone, Evangelos Tabakis, Jin-Gen Yang, and other students and colleagues.

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R. M. Dudley

Contents

1

Foundations; Set Theory

In constructing a building, the builders may well use different techniques and materials to lay the foundation than they use in the rest of the building. Likewise, almost every field of mathematics can be built on a foundation of axiomatic set theory. This foundation is accepted by most logicians and mathematicians concerned with foundations, but only a minority of mathematicians have the time or inclination to learn axiomatic set theory in detail.

To make another analogy, higher-level computer languages and programs written in them are built on a foundation of computer hardware and systems programs. How much the people who write high-level programs need to know about the hardware and operating systems will depend on the problem at hand.

In modern real analysis, set-theoretic questions are somewhat more to the fore than they are in most work in algebra, complex analysis, geometry, and applied mathematics. A relatively recent line of development in real analysis, "nonstandard analysis," allows, for example, positive numbers that are infinitely small but not zero. Nonstandard analysis depends even more heavily on the specifics of set theory than earlier developments in real analysis did.

This chapter will give only enough of an introduction to set theory to define some notation and concepts used in the rest of the book. In other words, this chapter presents mainly "naive" (as opposed to axiomatic) set theory. Appendix A gives a more detailed development of set theory, including a listing of axioms, but even there, the book will not enter into nonstandard analysis or develop enough set theory for it.

Many of the concepts defined in this chapter are used throughout mathematics and will, I hope, be familiar to most readers.

1.1. Definitions for Set Theory and the Real Number System

Definitions can serve at least two purposes. First, as in an ordinary dictionary, a definition can try to give insight, to convey an idea, or to explain a less familiar idea in terms of a more familiar one, but with no attempt to specify or exhaust completely the meaning of the word being defined. This kind of definition will be called *informal.* A *formal* definition, as in most of mathematics and parts of other sciences, may be quite precise, so that one can decide scientifically whether a statement about the term being defined is true or not. In a formal definition, a familiar term, such as a common unit of length or a number, may be defined in terms of a less familiar one. Most definitions in set theory are formal. Moreover, set theory aims to provide a coherent logical structure not only for itself but for just about all of mathematics. There is then a question of where to begin in giving definitions.

Informal dictionary definitions often consist of synonyms. Suppose, for example, that a dictionary simply defined "high" as "tall" and "tall" as "high." One of these definitions would be helpful to someone who knew one of the two words but not the other. But to an alien from outer space who was trying to learn English just by reading the dictionary, these definitions would be useless. This situation illustrates on the smallest scale the whole problem the alien would have, since all words in the dictionary are defined in terms of other words. To make a start, the alien would have to have some way of interpreting at least a few of the words in the dictionary other than by just looking them up.

In any case some words, such as the conjunctions "and," "or," and "but," are very familiar but hard to define as separate words. Instead, we might have rules that define the meanings of phrases containing conjunctions given the meanings of the words or subphrases connected by them.

At first thought, the most important of all definitions you might expect in set theory would be the definition of "set," but quite the contrary, just because the entire logical structure of mathematics reduces to or is defined in terms of this notion, it cannot necessarily be given a formal, precise definition. Instead, there are rules (axioms, rules of inference, etc.) which in effect provide the meaning of "set." A preliminary, informal definition of *set* would be "any collection of mathematical objects," but this notion will have to be clarified and adjusted as we go along.

The problem of defining *set* is similar in some ways to the problem of defining *number*. After several years of school, students "know" about the numbers 0, 1, 2, ..., in the sense that they know rules for operating with numbers. But many people might have a hard time saying exactly what a number is. Different people might give different definitions of the number 1, even though they completely agree on the rules of arithmetic.

In the late 19th century, mathematicians began to concern themselves with giving precise definitions of numbers. One approach is that beginning with 0, we can generate further integers by taking the "successor" or "next larger integer."

If 0 is defined, and a successor operation is defined, and the successor of any integer n is called n', then we have the sequence $0, 0', 0'', 0''', \ldots$. In terms of 0 and successors, we could then write down definitions of the usual integers. To do this I'll use an equals sign with a colon before it, ":=," to mean "equals by definition." For example, $1 := 0', 2 := 0'', 3 := 0''', 4 := 0''''$, and so on. These definitions are precise, as far as they go. One could produce a thick dictionary of numbers, equally precise (though not very useful) but still incomplete, since 0 and the successor operation are not formally defined. More of the structure of the number system can be provided by giving rules about 0 and successors. For example, one rule is that if $m' = n'$, then $m = n$.

Once there are enough rules to determine the structure of the nonnegative integers, then what is important is the structure rather than what the individual elements in the structure actually are.

In summary: if we want to be as precise as possible in building a rigorous logical structure for mathematics, then informal definitions cannot be part of the structure, although of course they can help to explain it. Instead, at least some basic notions must be left undefined. Axioms and other rules are given, and other notions are defined in terms of the basic ones.

Again, informally, a set is any collection of objects. In mathematics, the objects will be mathematical ones, such as numbers, points, vectors, or other sets. (In fact, from the set-theoretic viewpoint, all mathematical objects are sets of one kind or another.) If an object x is a member of a set y, this is written as "$x \in y$," sometimes also stated as "x belongs to y" or "x is in y." If S is a finite set, so that its members can be written as a finite list $x_1, \ldots, x_n$, then one writes $S = \{x_1, \ldots, x_n\}$. For example, $\{2, 3\}$ is the set whose only members are the numbers 2 and 3. The notion of membership, "$\in$," is also one of the few basic ones that are formally undefined.

A set can have just one member. Such a set, whose only member is x, is called $\{x\}$, read as "singleton x." In set theory a distinction is made between $\{x\}$ and x itself. For example if $x = \{1, 2\}$, then x has two members but $\{x\}$ only one.

A set A is *included* in a set B, or is a *subset* of B, written $A \subset B$, if and only if every member of A is also a member of B. An equivalent statement is that B *includes* A, written $B \supset A$. To say B *contains* x means $x \in B$. Many authors also say B contains A when $B \supset A$.

The phrase "if and only if" will sometimes be abbreviated "iff." For example, $A \subset B$ iff for all x, if $x \in A$, then $x \in B$.

One of the most important rules in set theory is called "extensionality." It says that if two sets A and B have the same members, so that for any object

$x, x \in A$ if and only if $x \in B$, or equivalently both $A \subset B$ and $B \subset A$, then the sets are equal, $A = B$. So, for example, $\{2, 3\} = \{3, 2\}$. The order in which the members happen to be listed makes no difference, as long as the members are the same. In a sense, extensionality is a definition of equality for sets. Another view, more common among set theorists, is that any two objects are equal if and only if they are identical. So "$\{2, 3\}$" and "$\{3, 2\}$" are two names of one and the same set.

Extensionality also contributes to an informal definition of *set*. A set is defined simply by what its members are—beyond that, structures and relationships between the members are irrelevant to the definition of the set.

Other than giving finite lists of members, the main way to define specific sets is to give a condition that the members satisfy. In notation, $\{x: \ldots\}$ means the set of all x such that.... For example, $\{x: (x-4)^2 = 4\} = \{2, 6\} = \{6, 2\}$.

In line with a general usage that a slash through a symbol means "not," as in $a \neq b$, meaning "a is not equal to b," the symbol "$\notin$" means "is not a member of." So $x \notin y$ means x is not a member of y, as in $3 \notin \{1, 2\}$.

Defining sets via conditions can lead to contradictions if one is not careful. For example, let $r = \{x: x \notin x\}$. Then $r \notin r$ implies $r \in r$ and conversely (Bertrand Russell's paradox). This paradox can be avoided by limiting the condition to some set. Thus $\{x \in A: \ldots x \ldots\}$ means "the set of all x in A such that $\ldots x \ldots$." As long as this form of definition is used when A is already known to be a set, new sets can be defined this way, and it turns out that no contradictions arise.

It might seem peculiar, anyhow, for a set to be a member of itself. It will be shown in Appendix A (Theorem A.1.9), from the axioms of set theory listed there, that no set is a member of itself. In this sense, the collection r of sets named in Russell's paradox is the collection of all sets, sometimes called the "universe" in set theory. Here the informal notion of set as any collection of objects is indeed imprecise. The axioms in Appendix A provide conditions under which certain collections are or are not sets. For example, the universe is not a set.

Very often in mathematics, one is working for a while inside a fixed set y. Then an expression such as $\{x: \ldots x \ldots\}$ is used to mean $\{x \in y: \ldots x \ldots\}$.

Now several operations in set theory will be defined. In cases where it may not be obvious that the objects named are sets, there are axioms which imply that they are (Appendix A).

There is a set, called $\emptyset$, the "empty set," which has no members. That is, for all x, $x \notin \emptyset$. This set is unique, by extensionality. If B is any set, then 2^B, also called the "power set" of B, is the set of all subsets of B. For example, if B has 3 members, then 2^B has $2^3 = 8$ members. Also, $2^{\emptyset} = \{\emptyset\} \neq \emptyset$.

$A \cap B$, called the intersection of A and B, is defined by $A \cap B := \{x \in A: x \in B\}$. In other words, $A \cap B$ is the set of all x which belong to both A and B. $A \cup B$, called the union of A and B, is a set such that for any x, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$ (or both). Also, $A \setminus B$ (read "A minus B") is the set of all x in A which are not in B, sometimes called the *relative complement* (of B in A). The *symmetric difference* $A \Delta B$ is defined as $(A \setminus B) \cup (B \setminus A)$.

$\mathbb{N}$ will denote the set of all nonnegative integers 0, 1, 2,.... (Formally, nonnegative integers are usually defined by defining 0 as the empty set $\emptyset$, 1 as $\{\emptyset\}$, and generally the successor operation mentioned above by $n' = n \cup \{n\}$, as is treated in more detail in Appendix A.)

Informally, an *ordered pair* consists of a pair of mathematical objects in a given order, such as $\langle x, y \rangle$, where x is called the "first member" and y the "second member" of the ordered pair $\langle x, y \rangle$. Ordered pairs satisfy the following axiom: for all x, y, u, and v, $\langle x, y \rangle = \langle u, v \rangle$ if and only if both $x = u$ and $y = v$. In an ordered pair $\langle x, y \rangle$ it may happen that $x = y$. Ordered pairs can be defined formally in terms of (unordered, ordinary) sets so that the axiom is satisfied; the usual way is to set $\langle x, y \rangle := \{\{x\}, \{x, y\}\}$ (as in Appendix A). Note that $\{\{x\}, \{x, y\}\} = \{\{y, x\}, \{x\}\}$ by extensionality.

One of the main ideas in all of mathematics is that of function. Informally, given sets D and E, a function f on D is defined by assigning to each x in D one (and only one!) member $f(x)$ of E. Formally, a *function* is defined as a set f of ordered pairs $\langle x, y \rangle$ such that for any x, y, and z, if $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$, then $y = z$. For example, $\{\langle 2, 4 \rangle, \langle -2, 4 \rangle\}$ is a function, but $\{\langle 4, 2 \rangle, \langle 4, -2 \rangle\}$ is not a function. A set of ordered pairs which is (formally) a function is, informally, called the *graph* of the function (as in the case $D = E = \mathbb{R}$, the set of real numbers).

The *domain*, *dom f*, of a function f is the set of all x such that for some y, $\langle x, y \rangle \in f$. Then y is uniquely determined, by definition of function, and it is called $f(x)$. The *range*, *ran f*, of f is the set of all y such that $f(x) = y$ for some x.

A function f with domain A and range included in a set B is said to be *defined on A* or *from A into B*. If the range of f equals B, then f is said to be *onto B*.

The symbol "$\mapsto$" is sometimes used to describe or define a function. A function f is written as "$x \mapsto f(x)$." For example, "$x \mapsto x^3$" or "$f: x \mapsto x^3$" means a function f such that $f(x) = x^3$ for all x (in the domain of f). To specify the domain, a related notation in common use is, for example, "$f: A \mapsto B$," which together with a more specific definition of f indicates that it is defined from A into B (but does not mean that $f(A) = B$; to distinguish the two related usages of $\mapsto$, A and B are written in capitals and members of them in small letters, such as x).

If X is any set and A any subset of X, the *indicator function* of A (on X) is the function defined by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

(Many mathematicians call this the *characteristic* function of A. In probability theory, "characteristic function" happens to mean a Fourier transform, to be treated in Chapter 9.)

A *sequence* is a function whose domain is either $\mathbb{N}$ or the set $\{1, 2, \ldots\}$ of all positive integers. A sequence f with $f(n) = x_n$ for all n is often written as $\{x_n\}_{n\geq 1}$ or the like.

Formally, every set is a set of sets (every member of a set is also a set). If a set is to be viewed, also informally, as consisting of sets, it is often called a family, class, or collection of sets. Let $\mathcal{V}$ be a family of sets. Then the *union* of $\mathcal{V}$ is defined by

$$\bigcup \mathcal{V} := \{x : x \in A \text{ for some } A \in \mathcal{V}\}.$$

Likewise, the *intersection* of a non-empty collection $\mathcal{V}$ is defined by

$$\bigcap \mathcal{V} := \{x : x \in A \text{ for all } A \in \mathcal{V}\}.$$

So for any two sets A and B, $\bigcup\{A, B\} = A \cup B$ and $\bigcap\{A, B\} = A \cap B$. Notations such as $\bigcup \mathcal{V}$ and $\bigcap \mathcal{V}$ are most used within set theory itself. In the rest of mathematics, unions and intersections of more than two sets are more often written with indices. If $\{A_n\}_{n\geq 1}$ is a sequence of sets, their union is written as

$$\bigcup_n A_n := \bigcup_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for some } n\}.$$

Likewise, their intersection is written as

$$\bigcap_{n\geq 1} A_n := \bigcap_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for all } n\}.$$

The union of finitely many sets $A_1, \ldots, A_n$ is written as

$$\bigcup_{1\leq i\leq n} A_i := \bigcup_{i=1}^{n} A_i := \{x : x \in A_i \text{ for some } i = 1, \ldots, n\},$$

and for intersections instead of unions, replace "some" by "all."

More generally, let I be any set, and suppose A is a function defined on I whose values are sets $A_i := A(i)$. Then the union of all these sets A_i is written

$$\bigcup_i A_i := \bigcup_{i \in I} A_i := \{x : x \in A_i \text{ for some } i\}.$$

A set I in such a situation is called an *index set*. This just means that it is the domain of the function $i \mapsto A_i$. The index set I can be omitted from the notation, as in the first expression above, if it is clear from the context what I is. Likewise, the intersection is written as

$$\bigcap_i A_i := \bigcap_{i \in I} A_i := \{x : x \in A_i \text{ for all } i \in I\}.$$

Here, usually, I is a non-empty set. There is an exception when the sets under discussion are all subsets of one given set, say X. Suppose $t \notin I$ and let $A_t := X$. Then replacing I by $I \cup \{t\}$ does not change $\bigcap_{i \in I} A_i$ if I is non-empty. In case I is empty, one can set $\bigcap_{i \in \emptyset} A_i = X$.

Two more symbols from mathematical logic are sometimes useful as abbreviations: $\forall$ means "for all" and $\exists$ means "there exists." For example, $(\forall x \in A)(\exists y \in B)\ldots$ means that for all x in A, there is a y in B such that....

Two sets A and B are called *disjoint* iff $A \cap B = \emptyset$. Sets A_i for $i \in I$ are called disjoint iff $A_i \cap A_j = \emptyset$ for all $i \neq j$ in I.

Next, some definitions will be given for different classes of numbers, leading up to a definition of real numbers. It is assumed that the reader is familiar with integers and rational numbers. A somewhat more detailed and formal development is given in Appendix A.4.

Recall that $\mathbb{N}$ is the set of all nonnegative integers $0, 1, 2, \ldots$, $\mathbb{Z}$ denotes the set of all integers $0, \pm 1, \pm 2, \ldots$, and $\mathbb{Q}$ is the set of all rational numbers m/n, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, and $n \neq 0$.

Real numbers can be defined in different ways. A familiar way is through decimal expansions: x is a real number if and only if $x = \pm y$, where $y = n + \sum_{j=1}^{\infty} d_j/10^j$, $n \in \mathbb{N}$, and each digit d_j is an integer from 0 to 9. But decimal expansions are not very convenient for proofs in analysis, and they are not unique for rational numbers of the form $m/10^k$ for $m \in \mathbb{Z}$, $m \neq 0$, and $k \in \mathbb{N}$. One can also define real numbers x in terms of more general sequences of rational numbers converging to x, as in the completion of metric spaces to be treated in §2.5.

The formal definition of real numbers to be used here will be by way of Dedekind cuts, as follows: A *cut* is a set $C \subset \mathbb{Q}$ such that $C \notin \emptyset$; $C \neq \mathbb{Q}$; whenever $q \in C$, if $r \in \mathbb{Q}$ and $r < q$ then $r \in C$, and there exists $s \in \mathbb{Q}$ with $s > q$ and $s \in C$.

Let $\mathbb{R}$ be the set of all real numbers; thus, formally, $\mathbb{R}$ is the set of all cuts. Informally, a one-to-one correspondence between real numbers x and cuts C, written $C = C_x$ or $x = x_C$, is given by $C_x = \{q \in \mathbb{Q}: q < x\}$.

The ordering $x \leq y$ for real numbers is defined simply in terms of cuts by $C_x \subset C_y$. A set E of real numbers is said to be *bounded above* with an *upper bound* y iff $x \leq y$ for all $x \in E$. Then y is called the *supremum* or *least upper bound* of E, written $y = \sup E$, iff it is an upper bound and $y \leq z$ for every upper bound z of E. A basic fact about $\mathbb{R}$ is that for every non-empty set $E \subset \mathbb{R}$ such that E is bounded above, the supremum $y = \sup E$ exists. This is easily proved by cuts: C_y is the union of the cuts C_x for all $x \in E$, as is shown in Theorem A.4.1 of Appendix A.

Similarly, a set F of real numbers is *bounded below* with a *lower bound* v if $v \leq x$ for all $x \in F$, and v is the *infimum* of F, $v = \inf F$, iff $t \leq v$ for every lower bound t of F. Every non-empty set F which is bounded below has an infimum, namely, the supremum of the lower bounds of F (which are a non-empty set, bounded above).

The maximum and minimum of two real numbers are defined by $\min(x, y) = x$ and $\max(x, y) = y$ if $x \leq y$; otherwise, $\min(x, y) = y$ and $\max(x, y) = x$.

For any real numbers $a \leq b$, let $[a, b] := \{x \in \mathbb{R}: a \leq x \leq b\}$.

For any two sets X and Y, their *Cartesian product*, written $X \times Y$, is defined as the set of all ordered pairs $\langle x, y\rangle$ for x in X and y in Y. The basic example of a Cartesian product is $\mathbb{R} \times \mathbb{R}$, which is also written as $\mathbb{R}^2$ (pronounced r-two, not r-squared), and called the *plane*.

Problems

1. Let $A := \{3, 4, 5\}$ and $B := \{5, 6, 7\}$. Evaluate: (a) $A \cup B$. (b) $A \cap B$. (c) $A \setminus B$. (d) $A \,\Delta\, B$.
2. Show that $\emptyset \neq \{\emptyset\}$ and $\{\emptyset\} \neq \{\{\emptyset\}\}$.
3. Which of the following three sets are equal? (a) $\{\{2, 3\}, \{4\}\}$; (b) $\{\{4\}, \{2, 3\}\}$; (c) $\{\{4\}, \{3, 2\}\}$.
4. Which of the following are functions? Why?
   (a) $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$.
   (b) $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 2, 1\rangle\}$.
   (c) $\{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 1, 2\rangle\}$.
   (d) $\{\langle x, y\rangle \in \mathbb{R}^2: x = y^2\}$.
   (e) $\{\langle x, y\rangle \in \mathbb{R}^2: y = x^2\}$.
5. For any relation V (that is, any set of ordered pairs), define the *domain* of