Introduction to Chaos

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Introduction

This book includes four chapters, and mainly introduces the contents of the basic computational techniques of linear differential equation, the qualitative or geometric approaches for planar differential systems, the method of computation and analysis of the chaotic system, and several methods of feedback control, backstepping control and generalized synchronization for chaotic systems.

This book presents an introduction to chaos in dynamical system and fundamental theories of ordinary differential equations. It will be of interest to advance undergraduates in mathematics and graduate students in engineering taking courses in dynamical systems, nonlinear dynamics, nonlinear systems as well as chaos.

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Preface

This book presents an introduction to chaos in dynamical system and fundamental theories of ordinary differential equations. Considering the fact that many readers have unclear concepts about chaotic dynamical systems and some mathematical concepts are also abstract, this book mainly focuses on the explanation of the systems' basic concepts, with detailed examples and diagrams. It also attaches great importance to the calculational and analytic methods of the chaos systems and the application of their theories. It will be of interest to advanced undergraduates in mathematics and graduate students in engineering taking courses in dynamical systems, nonlinear dynamics, nonlinear systems as well as chaos.

In chapter 1, the basic computational techniques of linear differential equation are discussed. For nonlinear differential equations, their general solutions are rarely obtained. Hence, the chapter 2 introduces the qualitative or geometric approaches for planar differential systems, whose materials mainly include equilibrium points, linearization, periodic orbits and Hamiltonian systems. It is the foundation to understand the chaos in dynamical systems. In chapter 3, the basic concepts of chaos are presented. The attractors, Lyapunov exponents, center manifolds, Hopf bifurcation, invariant algebraic surfaces and infinite singular points analysis of chaotic systems are shown. In its final chapter, several of feedback control, backstepping control and generalized synchronization for chaotic systems are reported. We hope this book will produce a very good introductory effect for beginners. Due to the shallow knowledge of the authors, the comments and opinions will be expected and welcomed from readers, experts and scholars by this email williamwangz@126.com.

The book's contents mainly derive from our research papers and reports over the recent years. In order to describe the basic theories and concepts vividly, the authors also draw on books written by other scholars and experts. Among them are Jack Hale's *Theory of Functional Differential equations*, Lawrence Perko's *Differential Equations and Dynamical Systems*, D.W. Jordan's *Nonlinear Ordinary Differential Equations*, R.C. Robinson's *Introduction to*

Dynamical Systems: Discrete and Continuous, K.T. Alligood's *Chaos: An Introduction to Dynamical Systems*, which have a greater influence on its formation. The authors appreciate their help very much. Meanwhile, I also want to express my gratitude to my tutor Lin Xiaolin (Shaanxi University of Science and Technology). It is him who leads me into the differential equation and the chaotic dynamic system.

The arrangement of its contents is schemed by Wang Zhen with its first chapter and second chapter written by Xi Xiaojian and the third chapter and the fourth one by Wang Zhen. It is Wang Zhen who shoulder the responsibility to correct and modify its knowledge points.

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Finally, we would like to acknowledge our students and colleagues who read and commented on various versions and parts of the manuscript. We also offer thanks to professor Wei Zhaochao (China University of Geosciences (Wuhan)), professor Wu Kuilin (Guizhou University) and especially our families for their patience and support while this book was being prepared. We wish to express our appreciation to Xidian University Press for giving us this opportunity to publish this book, and thank the editors for their help. It is my friend and colleague, Song Bingchang (Xijing University), who gives valuable advice on the book's written English and we appreciate his efforts very much.

Wang Zhen

Xijing Yuan, 2014

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CHAPTER 1

Computational Techniques of Linear Differential Equation

In this chapter, we introduce the basic computational methods for linear differential equation which includes the first order DE, second order DE and linear system.

1.1 Basic concepts

Let $U \subseteq \mathrm{R}^m$, $V \subseteq \mathrm{R}^n$ and $k \in \mathrm{N}$, then $C^k(U,V)$ denotes the set functions $U \to V$ having continuous derivates up to order k. In addition, we will abbreviate $C(U,V) = C^0(U,V)$ and $C^k(U) = C^k(U,R)$.

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a **differential equation**.

If the unknown function is a function of a single variable, the differential equation is called an **ordinary differential equation** (**ODE**), while if the unknown function is a function of several variables, the differential equation is called a **partial differential equation** (**PDE**).

Example 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x + 3 \tag{1}$$

$$\mathrm{e}^y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 \tag{2}$$

$$\frac{\partial^2 z}{\partial y^2}-4\frac{\partial^2 z}{\partial x^2}=0 \quad (3)$$

The **order** of a differential equation is the order of the highest derivative that appears in the equation.

More generally, the equation

$$F(t,u(t),u'(t),\cdots,u^{(n)}(t))=0 \quad (4)$$

is an ordinary differential equation of the nth order. Eq. (4) expresses a relation between the independent variable t and the value of the function u and its first n derivatives u, u', ..., $u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with y, y', ..., $y^{(n)}$ standing for $u(t),u'(t),\cdots,u^{(n)}(t)$. Thus Eq. (4) is written as

$$F(t,y,y',\cdots,y^{(n)})=0 \quad (5)$$

In general, a DE of order n can be written as Eq. (5). This is called a DE in **implicit form**. In many cases, we can solve for $y^{(n)}$ and write the DE in **explicit form**

$$y^{(n)}=f(t,y,y',\cdots,y^{(n-1)}) \quad (6)$$

A classical ODE is a relation of the form

$$F(t,x,x',\cdots,x^{(n)})=0 \quad (7)$$

For the unknown function $x\in C^n(J)$, $J\subseteq R$. Here $F\in C(U)$ with U an open subset of R^{n+2}. One frequently calls t the **independent variable** and x the **dependent variable**. A solution of Eq. (7) is a function $\varphi\in C^n(I)$, where $I\subseteq J$ is an interval, such that

$$F(t,\varphi(t),\varphi'(t),\cdots,\varphi(t)^{(n)})=0 \quad (8)$$

for all $t\in I$. This implicitly implies $(t,\varphi(t),\varphi'(t),\cdots,\varphi(t)^{(n)})\in U$ for all $t\in I$.

A function $\varphi(t)$ that when substituted for x in Eq. (7) satisfies the equation for all t in the interval I is called an **explicit solution** to the equation on I.

A relation $G(t,x)=0$ is said to be an **implicit solution** to Eq. (7) on the interval I if it defines one or more explicit solutions on I.

A DE of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t) \tag{9}$$

is called a **linear DE**. A DE which is not linear is called **nonlinear**. If $g(t) \equiv 0$, the linear DE is said to be **homogeneous**, otherwise it is said to be **non-homogeneous**. If all the $a_i(t)$ are constants, it is called a linear DE with constant coefficients. If the equation has the form Eq. (5), we can say that the Eq. (5) is linear if F is a linear function of the variables $y, y', \cdots, y^{(n)}$.

Example 2 Show that $\varphi(t) = t^{-2}$ is a solution of the DE $x' + 2tx^2 = 0$.

Solution Differentiating, we get $\varphi'(t) = -2t^{-3}$. So for all $t \neq 0$,

$$\varphi'(t) + 2t\varphi(t) = -2t^{-3} + 2t \cdot (t^{-2})^2 = -2t^{-3} + 2t^{-3} = 0$$

Thus $\varphi(t)$ is a solution to Example 2 on $(0, +\infty)$ and also on $(-\infty, 0)$.

Note By solving a DE, we mean finding all solutions to the DE (and determine the (largest) interval, called the interval of validity on which the DE is satisfied).

Remark (1) It is straightforward to check that $\varphi(t) \equiv 0$ and $\varphi_c(t) = (t^2 + c)^{-1}$ (where c is a constant) are solution to example 2. Moreover, it can be shown (using the existence and uniqueness theorem) that there is no more solution. Note that φ is a solution valid on R and that

(i) if $c > 0$, φ_c is a solution on R.

(ii) if $c = 0$, φ_0 is a solution on on $(0, +\infty)$ and also on $(-\infty, 0)$.

(iii) if $c = -p < 0$, φ_c is a solution on $(-\infty, -\sqrt{p})$, $(-\sqrt{p}, \sqrt{p})$ and $(\sqrt{p}, +\infty)$.

(2) Example 2 has infinitely solutions, namely the one parameter family $(\varphi_c)_{c \in R}$ together with the trivial solution.

(3) The graphs of any pair of solutions have no intersection (uniqueness of solution).

(4) If we add an additional condition to the solution, for example, $x(0) = -1$, then we get the unique solution $\varphi_{-1}(t) = (t^2 - 1)^{-1}$ and the solution is valid for $-1 < t < 1$.

We have seen in the previous section that the case of real valued functions if not enough and we should admit the case $x:\mathrm{R}\to\mathrm{R}^m$. This leads us to **systems of DDEs**

$$x_i^{(n)}=f_i(t,x,x',\cdots,x^{(n-1)}),\quad i=1,\cdots,m \tag{10}$$

An nth order **initial value problem (IVP)** is an nth order DE together with n initial conditions.

$$\begin{cases}F(t,x,x',\cdots,x^{(n)})=0\\ x(t_0)=x_0,x'(t_0)=x_1,\cdots,x^{(n-1)}(t_0)=x_{n-1}\end{cases} \tag{11}$$

A solution to the IVP is a solution to the DE on an interval containing t_0 (as interior point) that satisfies all the initial conditions.

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an **initial value problem**. The subsidiary conditions are **initial conditions**. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a **boundary value problem (BVP)** and the conditions are **boundary conditions**.

Example 3 The problem $x''+2x'=\mathrm{e}^t$, $x(\pi)=1$, $x'(\pi)=2$ is an IVP, because the two subsidiary conditions are given at $t=\pi$. The problem $x''+2x'=\mathrm{e}^t$, $x(0)=1$, $x(1)=1$ is a BVP, because the two subsidiary conditions are at the different values $t=0$ and $t=1$.

1.2 First order linear differential equation

1.2.1 Separable equation

A simple class of first order differential equation that can be solved using integration is the class of separable equation. There are equations

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

that can be rewritten to isolate the variables x and y (together with their differentials dx and dy) on opposite sides of the equations, as in

$$h(y)dy = g(x)dx \tag{2}$$

So the original right hand side $f(x, y)$ must have the factored form

$$f(x, y) = g(x)\frac{1}{h(y)} \tag{3}$$

More formally, we write $p(y) = \frac{1}{h(y)}$ and present the following definition.

If the right hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called separable. In other words, a first order equation is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y)$$

Method for solving separable equation to solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{4}$$

Multiply by dx and by $h(y) = \frac{1}{p(y)}$ to obtain

$$h(y)dy = g(x)dx$$

Then integrate both sides

$$\int h(y)dy = \int g(x)dx$$

$$H(y) = G(x) + C \tag{5}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Note Constant functions $y \equiv c$ such that $p(c) = 0$ are also solutions to (4), which may or may not be included in (5).

Example 1 Solve the $\dfrac{dy}{dx} = \dfrac{x-5}{y^2}$.

Solution Separate the variables and rewrite the equation in the form

$$y^2 dy = (x-5)dx$$

Integrating, we have

$$\int y^2 dy = \int (x-5)dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

and solving this last equation for y gives

$$y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{\frac{1}{3}}$$

Since C is a constant of integration that can be any real number, $3C$ can also be any real number. Replacing $3C$ by the single symbol K, we then have

$$y = \left(\frac{3x^2}{2} - 15x + K\right)^{\frac{1}{3}}$$

If we wish to abide by the custom of letting C represent an arbitrary constant, we can go one step further and use C instead of K in the final answer. This solution family is graphed in Figure 1.

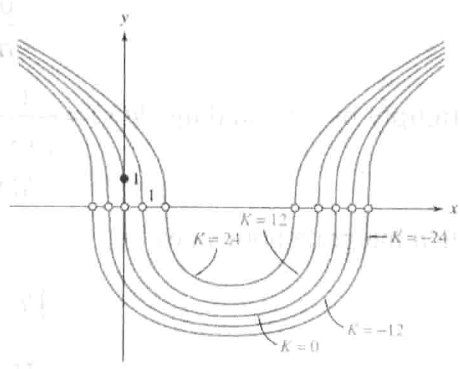

Figure 1 Family of solutions for Example 1

1.2.2 Linear equation

A first order linear DE is of the form

$$P(t)y' + Q(t)y = R(t), \quad a < t < b \tag{6}$$

where P, Q, R are functions on (a,b).

Suppose P never vanishes on (a,b) (that is $P(t) \neq 0$ for all $t \in (a,b)$), then dividing both sides by $P(t)$, (6) reduces to the standard form

$$y' + p(t)y = g(t), \quad a < t < b \tag{7}$$

where p, g are functions on (a,b). If $p, g \in C(a,b)$, then we can find solutions to the DE valid in whole internal (a,b).

To solve (7), we try to multiply both sides by a function $\mu = \mu(t)$ (which never vanishes on (a,b)), so that left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(t)y]$$

For this, we need

$$\mu'(t)y + \mu(t)y' = \mu(t)y' + \mu(t)p(t)y$$

That is

$$\mu'(t) = p(t)\mu(t)$$

By inspection, we see that $\mu(t) = \mathrm{e}^{H(t)}$ is a suitable candidate, where H is an antiderivative for p. The existence of H is guaranteed by the continuity of p, for example, fix $t_0 \in (a,b)$, then the function H given by $H(t) = \int_{t_0}^{t} p(s)\mathrm{d}s$ is an antiderivative for p.

Now multiplying (7) by the above $\mu(t)$, we get

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t) \tag{8}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(t)y] = \mu(t)g(t)$$

Integrate and then divide both sides by $\mu(t)$, we obtain

$$y(t)=\frac{1}{\mu(t)}\int\mu(t)g(t)\mathrm{d}t$$

Since $\mu(t)$ never vanishes on (a,b), solutions to (8) are also solutions to (7).

Theorem 1 The general solution for the DE

$$y'+p(t)y=g(t)$$

where $p,g\in C(a,b)$, is

$$y(t)=\mathrm{e}^{-H(t)}[\int\mathrm{e}^{H(t)}g(t)\mathrm{d}t]$$

where H is an antiderivative for p on (a,b).

The function $\mu(t)=\mathrm{e}^{H(t)}$ is called an integrating factor and the method is called the integrating factor method.

Remark Any non-zero multiple of μ is also an integrating factor. Thus it doesn't matter which antiderivative we take and we may write $\mu(t)=\mathrm{e}^{\int p(t)\mathrm{d}t}$.

Example 2 Find the general solution to the DE

$$y'+2ty=t$$

Solution The functions p and g, where $p(t)=2t$ and $g(t)=t$, are continuous on R. An antiderivative H for p is $H(t)=t^2$. Multiplying both sides of the DE by $\mu(t)=\mathrm{e}^{H(t)}$ yields

$$\mathrm{e}^{t^2}y'+2t\mathrm{e}^{t^2}y=t\mathrm{e}^{t^2}$$

Integrating both sides, we get

$$\mathrm{e}^{t^2}y=\int t\mathrm{e}^{t^2}\mathrm{d}t=\frac{1}{2}\mathrm{e}^{t^2}+C$$

The general solution is

$$y(t)=\frac{1}{2}+C\mathrm{e}^{-t^2},-\infty<t<\infty$$

Note that in the general solution given in theorem 1, there is a constant of integration in the integral $\int e^{H(t)} g(t) dt$. This constant can be determined if an initial condition is specified.

Theorem 2 The initial value problem (IVP)

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

where $p, g \in C(a,b)$, $t_0 \in (a,b)$ and $y_0 \in R$, has a unique solution valid on (a,b).

Proof Taking $H(t) = \int_{t_0}^{t} p(s) ds$ as an antiderivative for p, the general solution to the DE can be written as

$$y(t) = e^{-\int_{t_0}^{t} p(s) ds} \left[\int_{t_0}^{t} g(u) e^{\int_{t_0}^{u} p(s) ds} du + C \right]$$

Putting $y(t_0) = y_0$, we get $C = y_0$. Thus the unique solution is

$$y(t) = e^{-\int_{t_0}^{t} p(s) ds} \left[\int_{t_0}^{t} g(u) e^{\int_{t_0}^{u} p(s) ds} du + y_0 \right]$$

Example 3 Solve the following IVP

$$\begin{cases} ty' + 2y = 2t^2 + t \\ y(1) = 3 \end{cases}$$

and determine the (largest) interval on which the DE is satisfied.

Solution First we rewrite the DE as

$$y' + \frac{2}{t} y = 2t + 1 \tag{9}$$

The functions p and g, where $p(t) = -\frac{2}{t}$ and $g(t) = 2t + 1$, are continuous on