Regularized Semigroups and Non-Elliptic Differential Operators

（正则半群和非椭圆微分算子）

Zheng Quan, Li Miao

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Preface

This book gives a systematic treatment of the basic theory of regularized semigroups of bounded linear operators on Banach spaces, and of how such a theory may be applied to non-elliptic differential operators. It takes into account the development of the theory during the last twenty years, after the publication of the monograph [45] by deLaubenfels in 1993.

It is well known that the most important application of semigroups of operators is found in partial differential operators (PDOs) and the associated initial value problems(IVPs). However, the applications of classical semigroups of operators, C_0-semigroups, to PDOs are limited. A typical example is that the Schrödinger operator $i\Delta$ does not generate a C_0-semigroup on $L^p(p \neq 2)$[103].

Two important generalizations of C_0-semigroups, regularized semigroups and integrated semigroups, have received much attention since 1987 (see [45] and the references therein). The main reason is that their applications to elliptic and non-elliptic PDOs. In the first several years that these two classes of semigroups were introduced, there seemed to be a common bias that integrated semigroups were more useful than regularized semigroups. Over the last two decades, people have got some further understanding on the importance of applications of regularized semigroups to non-elliptic PDOs. Here are some reasons why regularized semigroups have advantage over integrated semigroups:

(a) The resolvent set of the generator of an integrated semigroup must be non-empty, even in the case of local integrated semigroups [207]. In contrast, the resolvent set of the generator of a regularized semigroup may be empty. There are many non-elliptic PDOs whose resolvent sets are empty (see, for example, [177]) thus they cannot generate integrated semigroups.

(b) Regularized semigroups keep more algebraic properties than integrated semigroups, and thus are easier to handle. An example is that regularized evolution families can be defined and applied to PDOs with time-dependent coefficients, see Chapter 6. However, it is difficult to define integrated evolution families. Another example is that Schrödinger operators with suitable potentials on $L^p(\mathbb{R}^n)(p \neq 2)$ can be treated conveniently by regularized semigroups [26, 169], while it is unpleasant to treat Schrödinger operators even without potentials by integrated semigroups [71].

(c) The classical solutions of abstract Cauchy problems associated with the generator of a regularized semigroup can be represented more naturally by the regularized semigroup. However, one has to meet with fractional powers for a representation of the solution via a fractional integrated semigroup, see Chapter 3. It is known that dealing with fractional powers of PDOs, even strongly elliptic PDOs, is a difficult and complex task [122]. In addition, it is hard to determine the behavior of the classical solution by looking at the integrated semigroups.

(d) The choice of the regularizing operator is flexible. For a concrete PDO with constant coefficients satisfying some conditions there are different ways to choose a regularizing operator C such that this PDO generates a C-regularized semigroup. It is possible that a better choice of the regularizing operator yields a bigger initial value space for the corresponding IVP than that one gets by integrated semigroups approach.

The book is organized as follows. The first two chapters provide a systematic introduction on the theory of regularized semigroups. Chapter 3 gives a short account of the theory of integrated semigroups, and the relationship between regularized semigroups and integrated semigroups. Chapters 4 and 5 treat PDOs that generate regularized semigroups or integrated semigroups, with an emphasis on why the regularized semigroup is a more appropriate tool for non-elliptic PDOs. Chapter 6 is devoted to the construction of regularized evolution family for PDOs with time-dependent coefficients, while Chapter 7 gives the applications of regularized semigroups to systems of partial differential equations including parabolic, correct and hyperbolic systems. Chapter 8 gives the recent development of the applications of regularized semigroups to Schrödinger operators of higher orders with the help of oscillatory integral theory.

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Contents

Chapter 1

Regularized Semigroups

We present in §1.1 the definitions and basic properties of regularized semigroups. Some generation theorems for regularized semigroups are provided in §1.2. The relations between regularized semigroups and C_0-semigroups via interpolation and extrapolation spaces are given in §1.3. We also consider different classes of regularized semigroups, such as contractive, norm continuous, differentiable, analytic and almost periodic regularized semigroups in §1.4. Finally, we will give the relation between regularized semigroups and the solutions of abstract Cauchy problems.

1.1 Definitions and properties

We start with the definition and basic properties of regularized semigroups. Let X be a Banach space, $B(X)$ the space of all bounded linear operators on X, and C an injective operator in $B(X)$. Moreover, the standard symbols are listed at the end of this book.

Definition 1.1.1 A strongly continuous family $T\colon [0, \infty) \to B(X)$ is called a C- *regularized semigroup* if $T(0) = C$ and $T(t+s)C = T(t)T(s)$ for $t, s \geqslant 0$. Its *generator* A is defined by

$$Ax = C^{-1}\Big(\lim_{t\downarrow 0}\frac{T(t)x - Cx}{t}\Big)$$

with maximal domain, i.e., $D(A) = \{x \in X : \text{the above limit exists and is in } R(C)$ (the range of $C)\}$.

It is obvious that if $C = I$ (the identity), then $\{T(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup (C_0-semigroup, in short). However, the following examples show the difference between regularized semigroups and C_0-semigroups.

Examples 1.1.2 (a) Let $Af(x) = xf(x)$ with maximal domain on $L^2(\mathbb{R})$. Then A is an unbounded self-adjoint operator in $L^2(\mathbb{R})$ and generates the e^{-A^2}-regularized semigroup given by $T(t) = e^{tA-A^2}$ for $t \geqslant 0$. A short calculation shows that $\|T(t)\| = e^{t^2/4}$ $(t \geqslant 0)$, this means that $\{T(t)\}_{t\geqslant 0}$ is not exponentially bounded.

(b) Let $X = C_0[0,\infty) := \{f \in C[0,\infty) : \ f(x) \to 0 \text{ as } x \to \infty\}$, $Af = -f'$ with $D(A) = \{f \in X \cap C^1[0,\infty) : \ f(0) = 0\}$. Then $(0,\infty) \subset \rho(A)$, and A generates the $R(1,A)$-regularized semigroup $\{R(1,A)S(t)\}_{t\geqslant 0}$, where

$$(S(t)f)(x) = \begin{cases} f(x-t), & \text{if } x - t \geqslant 0, \\ 0, & \text{if } x - t < 0. \end{cases}$$

But $\overline{D(A)} = \{f \in X : \ f(0) = 0\}$, and thus A is not densely defined on X.

The basic properties of C-regularized semigroup and its generator are collected in the following.

Theorem 1.1.3 *Let A be the generator of C-regularized semigroup $\{T(t)\}_{t\geqslant 0}$. Then the following assertions hold.*

(a) *If $f \in C^1([0,\infty), X)$, then $\int_0^t T(s)f(s)ds \in D(A)$ and*

$$A\int_0^t T(s)f(s)ds = T(t)f(t) - Cf(0) - \int_0^t T(s)f'(s)ds \quad (t \geqslant 0). \tag{1.1.1}$$

Particularly,

$$A\int_0^t T(s)xds = T(t)x - Cx \quad (x \in X, \ t \geqslant 0). \tag{1.1.2}$$

(b) *For $x \in D(A)$ and $t \geqslant 0$, $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$. Consequently,*

$$T(t)x - Cx = \int_0^t T(s)Axds \quad (x \in D(A), \ t \geqslant 0). \tag{1.1.3}$$

(c) *$D(A) = \{x \in X :$ there exists $y \in X$ such that $T(t)x - Cx = \int_0^t T(s)yds$ for $t \geqslant 0\}$ and $Ax = y$.*

(d) *A is closed, $R(C) \subseteq \overline{D(A)}$ and $A = C^{-1}AC$.*

(e) *$\{T(t)\}_{t\geqslant 0}$ is a unique C-regularized semigroup generated by A.*

(f) *If $\tilde{C}$ is an injective operator in $B(X)$ and $\tilde{C}T(t) = T(t)\tilde{C}$ for $t \geqslant 0$, then $\{\tilde{C}T(t)\}_{t\geqslant 0}$ is a $\tilde{C}C$-regularized semigroup generated by A.*

Proof We first note that for $t \geqslant 0$, $CT(t) = T(t)C$ and, by the resonance theorem, $\sup\{\|T(s)\| : 0 \leqslant s \leqslant t\} < \infty$.

(a) For $t \geqslant 0$, a simple calculation leads to

$$\begin{aligned}
&\frac{1}{h}(T(h)-C)\int_0^t T(s)f(s)ds \\
&=\frac{1}{h}\int_t^{t+h} T(s)f(s-h)ds-\frac{1}{h}\int_0^h CT(s)f(s)ds \\
&-\frac{1}{h}\int_h^t CT(s)(f(s)-f(s-h))ds \\
&\to CT(t)f(t)-C^2f(0)-\int_0^t CT(s)f'(s)ds,
\end{aligned}$$

as $h \to 0$, which implies (a).

(b) Fix $x \in D(A)$ and $t \geqslant 0$. For $h > 0$,

$$\frac{1}{h}(CT(t+h)x-CT(t)x)=\frac{1}{h}(T(h)-C)T(t)x=T(t)\frac{1}{h}(T(h)x-Cx).$$

Letting $h \downarrow 0$ we obtain that $T(t)x \in D(A)$ and

$$\frac{d^+}{dt}CT(t)x=CAT(t)x=CT(t)Ax.$$

Since $CT(\cdot)Ax \in C([0,\infty),X)$, this gives $\frac{d}{dt}CT(t)x=\frac{d^+}{dt}CT(t)x$, and thus the claim is easily deduced from the assumptions on C.

(c) Since C is injective, it is clear that y is uniquely determined by x. By (1.1.3) it remains to show that if $T(t)x-Cx=\int_0^t T(s)yds$ for $t \geqslant 0$, then $x \in D(A)$. This follows from

$$\frac{1}{t}(T(t)x-Cx)=\frac{1}{t}\int_0^t T(s)yds \to Cy,$$

as $t \to 0$.

(d) Taking a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ and $Ax_n \to y$. Then, by (1.1.3), for $t \geqslant 0$,

$$T(t)x-Cx=\lim_{n\to\infty}(T(t)x_n-Cx_n)=\int_0^t T(s)yds.$$

Thus, by (c), $x \in D(A)$ and $Ax=y$, i.e., A is closed. For $x \in X$,

$$D(A) \ni n\int_0^{1/n} T(s)xds \to Cx \quad (n \to \infty)$$

and hence $R(C) \subseteq \overline{D(A)}$. By (b) one sees that $CA \subseteq AC$, which implies that $A \subseteq C^{-1}AC$. Conversely, for $x \in D(C^{-1}AC)$, let $y = C^{-1}ACx$. Then, by the closeness of C^{-1} and (1.1.3),

$$\int_0^t T(s)yds=C^{-1}\int_0^t T(s)ACxds=T(t)x-Cx \quad (t \geqslant 0).$$

Thus it follows from (c) that $x \in D(A)$, as desired.

(e) If $\{S(t)\}_{t\geqslant 0}$ is also a C-regularized semigroup generated by A, then by (b) and (1.1.2),

$$\begin{aligned}\frac{d}{ds}T(t-s)\int_0^s S(r)xdr &= T(t-s)\Big(S(s)x - A\int_0^s S(r)xdr\Big)\\ &= T(t-s)Cx \quad (x\in X,\ t\geqslant s\geqslant 0).\end{aligned}$$

Integrating this from 0 to t we obtain

$$C\int_0^t T(r)xdr = C\int_0^t S(r)xdr \quad (x\in X,\ t\geqslant 0).$$

Since C is injective, $T(t) = S(t)$ for $t \geqslant 0$.

(f) It is clear that $\{\tilde{C}T(t)\}_{t\geqslant 0}$ is a $\tilde{C}C$-regularized semigroup. Let $\tilde{A}$ be its generator. Since $\tilde{C} \in B(X)$, it easily follows that $A \subseteq \tilde{A}$. If $x \in D(\tilde{A})$, then by (1.1.3),

$$\tilde{C}T(t)x - \tilde{C}Cx = \int_0^t \tilde{C}T(s)\tilde{A}xds = \tilde{C}\int_0^t T(s)\tilde{A}xds \quad (t\geqslant 0).$$

Operating with $\tilde{C}^{-1}$ on both sides and using (c) yield that $x \in D(A)$, as desired. □

The assertion (f) shows that the regularized semigroups generated by A can possess different regularizing operators C. So, in the application, it is remarkable how to choose an optimal C; that is, the operator C such that $R(C)$ is as large as possible.

Definition 1.1.4 A C-regularized semigroup $\{T(t)\}_{t\geqslant 0}$ is *exponentially bounded* if there exist constants $M \geqslant 0$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$ for $t \geqslant 0$. In this case, we write $(A, T(\cdot)) \in G(M, \omega, C)$ or $A \in G(M, \omega, C)$, where A is its generator. Also, set $G(\omega, C) = \cup_{M\geqslant 0} G(M, \omega, C)$ and $G(C) = \cup_{\omega\in\mathbb{R}} G(\omega, C)$.

Let A be a linear operator on X. The *C-resolvent set* of A is $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)\}$, and the *$C$-resolvent* of A is $R_C(\lambda, A) := (\lambda - A)^{-1}C$ for $\lambda \in \rho_C(A)$.

Here are some simple facts related with C-resolvents and the operator $C^{-1}AC$.

Proposition 1.1.5 (a) *If A is closed, then $R_C(\lambda, A) \in B(X)$ for $\lambda \in \rho_C(A)$. Conversely, if $R_C(\lambda, A) \in B(X)$ for some $\lambda \in \rho_C(A)$, then $C^{-1}AC$ is closed. Moreover, $C^{-1}AC$ is closed if A is closed.*

(b) *If $R_C(\lambda, A) = R_C(\lambda, \tilde{A})$ for some $\lambda \in \rho_C(A) \cap \rho_C(\tilde{A})$, then $C^{-1}AC = C^{-1}\tilde{A}C$. Moreover, $CD(C^{-1}AC) \subseteq R(R_C(\lambda, A))$ for $\lambda \in \rho_C(A)$.*

(c) *$CA \subseteq AC$ if and only if $C(\lambda - A)^{-1} \subseteq R_C(\lambda, A)$ for $\lambda \in \rho_C(A)$. Moreover,*

$$R_C(\lambda, A) - R_C(\mu, A) = (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}C \tag{1.1.4}$$

for λ, $\mu \in \rho_C(A)$.

(d) *If A is closed and $CA \subseteq AC$, then $C(D(A)) = R(R_C(\lambda, A))$ if and only if $\lambda \in \rho(A)$. In particular, $A = C^{-1}AC$ if $\rho(A) \neq \varnothing$.*

(e) *If $\tilde{A} \subseteq A$ and $A = C^{-1}AC$, then $C^{-1}\tilde{A}C = A$ if and only if $C(D(A)) \subseteq D(\tilde{A})$.*

Proof (a) If A is closed, it is easy to verity that for $\lambda \in \rho_C(A)$, $R_C(\lambda, A)$ is closed. Thus by the closed graph theorem, $R_C(\lambda, A) \in B(X)$. Conversely, let $D(C^{-1}AC) \ni x_n \to x$ and $C^{-1}ACx_n \to y$. Then $Cx_n \to Cx$ and

$$Cx_n = R_C(\lambda, A)C^{-1}(\lambda - A)Cx_n \to R_C(\lambda, A)(\lambda x - y).$$

Consequently, $Cx = R_C(\lambda, A)(\lambda x - y)$. It thus follows that $x \in D(C^{-1}AC)$ and $C^{-1}ACx = y$, i.e., $C^{-1}AC$ is closed. The rest follows easily by the definition of closed operators.

(b) For $x \in D(C^{-1}AC)$, let $y = \lambda x - C^{-1}ACx$. Then $Cx = R_C(\lambda, A)y = R_C(\lambda, \tilde{A})y$, which implies that $y = \lambda x - C^{-1}\tilde{A}Cx$, and thus $C^{-1}AC \subseteq C^{-1}\tilde{A}C$. The same argument gives the converse.

(c) This is easy to verify.

(d) From the assumption we deduce that $R(\lambda - A) = X$ if and only if $(\lambda - A)C(D(A)) = R(C)$. This fact together with the closed graph theorem leads to the desired assertion.

(e) It is clear that $C^{-1}\tilde{A}C \subseteq C^{-1}AC = A$. If $x \in D(A)$, then our assumption implies that $Cx \in D(\tilde{A})$ and $\tilde{A}Cx = ACx = CAx \in R(C)$. Thus, $x \in D(C^{-1}\tilde{A}C)$. The converse is easy from the definition of $D(C^{-1}\tilde{A}C)$. □

In the case that an extension of A generates a C-regularized semigroup, the assertion (e) can be used to check whether $C^{-1}AC$ is the generator.

The basic properties of exponentially bounded regularized semigroups is as follows.

Proposition 1.1.6 *Let $(A, T(\cdot)) \in G(M, \omega, C)$. Then for* $\text{Re}\lambda > \omega$ *and* $n \in \mathbb{N}$, *we have $\lambda \in \rho_C(A)$, $R(C) \subseteq R((\lambda - A)^n)$ and*

$$(\lambda - A)^{-n}Cx = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}T(t)xdt \quad (x \in X). \qquad (1.1.5)$$

Consequently,

(a) $\|(\lambda - A)^{-n}C\| \leqslant M(\text{Re}\lambda - \omega)^{-n}$ *for* $\text{Re}\lambda > \omega$ *and* $n \in \mathbb{N}$,

(b) $R_C(\cdot, A) : \{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\} \to B(X)$ *is analytic, and*

(c) $\lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1}x = x$ *for* $x \in R(C)$.

Proof For $\text{Re}\lambda > \omega$, define $R_\lambda \in B(X)$ by

$$R_\lambda x = \int_0^\infty e^{-\lambda t}T(t)xdt \quad (x \in X).$$

Then it follows from (1.1.1) that for $x \in X$,

$$(\lambda - A)\int_0^n e^{-\lambda t}T(t)xdt = Cx - e^{-\lambda n}T(n)x \to Cx \quad (n \to \infty).$$

Since A is closed, we have $R_\lambda x \in D(A)$ and $(\lambda - A)R_\lambda x = Cx$. Also, by Theorem 1.1.3(b), $T(t)A \subseteq AT(t)$ for $t \geqslant 0$, and thus $R_\lambda A \subseteq AR_\lambda$. Combining these assertions yields that $\lambda \in \rho_C(A)$ and $R_\lambda = R_C(\lambda, A)$. Now, by induction on n we obtain $R(C) \subseteq R((\lambda - A)^n)$ and (1.1.5). Moreover, a direct computation leads to (a), while the consequences (b) and (c) follow from Theorem A.5, immediately. □

We now turn to regularized groups.

Definition 1.1.7 A strongly continuous family $T : \mathbb{R} \to B(X)$ is called a *C-regularized group* if $T(0) = C$ and $T(t+s)C = T(t)T(s)$ for $t, s \in \mathbb{R}$. Its generator A is defined by

$$Ax = C^{-1}\lim_{t\to 0} t^{-1}(T(t)x - Cx)$$

with maximal domain. Moreover, a C-regularized group $\{T(t)\}_{t\in\mathbb{R}}$ is entire if it can be extended to an entire $B(X)$-valued function $\{T(t)\}_{t\in\mathbb{C}}$.

It is clear that the e^{-A^2}-regularized semigroup defined in Example 1.1.2(a) can be extended to a e^{-A^2}-regularized group, and an entire e^{-A^2}-regularized group as well. In general case, we have the following relationship between regularized semigroups and regularized groups.

Theorem 1.1.8 *The following are equivalent.*

(a) *A generates a C-regularized group $\{T(t)\}_{t\in\mathbb{R}}$.*

(b) *A and $-A$ generate C-regularized semigroups $\{T_+(t)\}_{t\geqslant 0}$ and $\{T_-(t)\}_{t\geqslant 0}$, respectively.*

Furthermore, $T(t) = T_+(t)$ and $T(-t) = T_-(t)$ for $t \geqslant 0$.

Proof (a) ⇒ (b). Set $T_\pm(t) = T(\pm t)$ $(t \geqslant 0)$. Then both $T_+(t)$ and $T_-(t)$ are C-regularized semigroups. Suppose their generators are A_+ and A_-, respectively. By the definition of generators, $A \subseteq A_+$. Conversely, if $x \in D(A_+)$, then for $t \in \mathbb{R}$,

$$\begin{aligned}\frac{1}{h}(T(t+h)x - T(t)x) &= C^{-1}T(t)\frac{1}{h}(T_+(h)x - Cx)\\ &\to T(t)A_+x \quad (h \downarrow 0).\end{aligned}$$

Thus $\frac{d}{dt}T(t)x = T(t)A_+x$. Taking $t = 0$, one has $x \in D(A)$, and so $A_+ = A$. Similarly, $A_- = -A$.

(b) ⇒ (a). For $x \in X$ and $t \geqslant 0$, it follows from Theorem 1.1.3(a) and (b) that

$$\frac{d}{dt}T_+(t)\int_0^t T_-(s)xds = T_+(t)\Big(A\int_0^t T_-(s)xds + T_-(t)x\Big) = T_+(t)Cx.$$

Integrating this we get

$$T_+(t)\int_0^t T_-(s)xds = \int_0^t T_+(s)Cxds.$$

Operating with A on both sides and together with (1.1.2) we get $T_+(t)T_-(t)x = C^2x$. Similarly, $T_-(t)T_+(t)x = C^2x$. Define $T(\pm t) = T_\pm(t)$ for $t \geqslant 0$. Then, it is easy to check that $\{T(t)\}_{t\in\mathbb{R}}$ is a C-regularized group. Suppose $\tilde{A}$ is its generator. Then, by the proof of (a)$\Rightarrow$(b), $\tilde{A} = A$. □

It is not hard to show that (a) is also equivalent to that A generates a C-regularized semigroup $\{T(t)\}_{t\geqslant 0}$ satisfying $0 \in \rho_{C^2}(T(t))$ $(t > 0)$ and $T(\cdot)^{-1}C^2x \in C([0,\infty), X)$ $(x \in X)$.

1.2 Generation theorems

This section is concerned with some generation theorems of exponentially bounded regularized semigroups. We first characterize generators of C-regularized semigroups in terms of Laplace transforms.

Theorem 1.2.1 *Let $M \geqslant 0$, $\omega \in \mathbb{R}$. Then the following are equivalent.*

(a) $A \in G(M, \omega, C)$.

(b) $A = C^{-1}AC$, $(\omega, \infty) \subset \rho_C(A)$ *and there exists a strongly continuous family* $T : [0,\infty) \to B(X)$ *satisfying* $\|T(t)\| \leqslant Me^{\omega t}$ *for* $t \geqslant 0$ *such that*

$$R_C(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt \quad (\lambda > \omega,\ x \in X). \tag{1.2.1}$$

Furthermore, $\{T(t)\}_{t\geqslant 0}$ *is the C-regularized semigroup generated by A.*

Proof (a) $\Rightarrow$ (b). This follows from Theorem 1.1.3(d) and Proposition 1.1.6.

(b) $\Rightarrow$ (a). By (1.2.1) and Proposition 1.1.5(c), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty e^{-\mu t}e^{-\lambda s}T(t+s)Cxdsdt \\
&= \int_0^\infty e^{-(\mu-\lambda)t}\Big(R_C(\lambda, A)Cx - \int_0^t e^{-\lambda\tau}T(\tau)Cxd\tau\Big)dt \\
&= \frac{1}{\mu-\lambda}(R_C(\lambda, A)Cx - R_C(\mu, A)Cx) \\
&= R_C(\mu, A)R_C(\lambda, A)x \\
&= \int_0^\infty \int_0^\infty e^{-\mu t}e^{-\lambda s}T(t)T(s)xdsdt \quad (\mu > \lambda > \omega,\ x \in X).
\end{aligned}$$

So, the uniqueness of Laplace transforms yields that $T(t+s)C = T(t)T(s)$ for $t, s \geqslant 0$. To show that $T(0) = C$, we note that $T(0)(T(0) - C) = 0$, and thus it suffices to prove the injectivity of $T(0)$. If $T(0)x = 0$, then $T(t)Cx = T(t)T(0)x = 0$. By (1.2.1), $R_C(\lambda, A)Cx = 0$ for $\lambda > \omega$. This implies that $x = 0$, as desired. Hence $\{T(t)\}_{t\geqslant 0}$ is a C-regularized semigroup. By Proposition 1.1.5(b), A is its generator. □

If, in (b), the condition "$A = C^{-1}AC$" is replaced by "$CA \subseteq AC$", then (b) implies that $C^{-1}AC \in G(M, \omega, C)$. This note is also suitable to generation theorems that appear later.

As an application of Theorem 1.2.1, we give the following.

Proposition 1.2.2 *Let $(A, T(\cdot)) \in G(M, \omega, C)$. Then for $a > \omega$, $((a - A)^{-1}, S(\cdot)) \in G(M, (a-\omega)^{-1}, C)$, where*

$$S(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(a - A)^{-n}C \quad (t \geqslant 0). \tag{1.2.2}$$

In particular, $A^{-1} \in G(C)$ if $\omega < 0$.

Proof For fixed $a > \omega$, let $\tilde{A} = (a - A)^{-1}$ and $\tilde{\omega} = (a - \omega)^{-1}$. We first show that $\tilde{A} = C^{-1}\tilde{A}C$. Since $A \subseteq C^{-1}AC$, it is easy to verify that $\tilde{A} \subseteq C^{-1}\tilde{A}C$. Conversely, let $x \in D(C^{-1}\tilde{A}C)$ and $y = C^{-1}\tilde{A}Cx$. Then $x = ay - C^{-1}ACy = (a - A)y \in D(\tilde{A})$.

Next, let $\lambda > \tilde{\omega}$. Then $a - \frac{1}{\lambda} > \omega$. Since $\lambda - \tilde{A} = \lambda(a - \frac{1}{\lambda} - A)(a - A)^{-1}$, we see that $\lambda - \tilde{A}$ is injective and $R(C) \subseteq R(\lambda - \tilde{A})$. Hence $(\tilde{\omega}, \infty) \subseteq \rho_C(\tilde{A})$.

Finally, (1.1.5) implies that $\|\tilde{A}^n C\| \leqslant M\tilde{\omega}^n$ for $n \in \mathbb{N}$, and thus we may define $\{S(t)\}_{t\geqslant 0}$ by (1.2.2). It is clear that $\{S(t)\}_{t\geqslant 0}$ is strongly continuous, with $\|S(t)\| \leqslant Me^{\tilde{\omega}t}$ for $t \geqslant 0$. Since $\tilde{A} = C^{-1}\tilde{A}C$ and $\tilde{A}C \in B(X)$, it follows that $\tilde{A}$ is closed, and thus, by (1.2.2),

$$\begin{aligned}\int_0^{\infty} e^{-\lambda t} S(t)x dt &= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{t^n}{n!} e^{-\lambda t} \tilde{A}^n Cx dt \\ &= \sum_{n=0}^{\infty} \lambda^{-n-1} \tilde{A}^n Cx = (\lambda - \tilde{A})^{-1} Cx\end{aligned}$$

for $\lambda > \tilde{\omega}$ and $x \in X$. Now, the claim follows from Theorem 1.2.1. □

It is well known that an important generation theorem for C_0-semigroups is the Hille-Yosida-Phillips-Miyadera theorem. Now we establish an analogous result for regularized semigroups.

Theorem 1.2.3 *Let $M \geqslant 0$, $\omega \in \mathbb{R}$. Then the following statements are equivalent.*

(a) $A = C^{-1}AC$, $(\omega, \infty) \subseteq \rho_C(A)$, $R(C) \subseteq R((\lambda - A)^n)$ *and*

$$\|(\lambda - \omega)^n (\lambda - A)^{-n} C\| \leqslant M \quad (\lambda > \omega,\ n \in \mathbb{N}). \tag{1.2.3}$$

(b) $(A, T(\cdot)) \in G(R_C(a, A))$ *for some* $a \in \rho_C(A)$ *and* $\{e^{-\tilde{\omega}t}T(t)\}_{t \geqslant 0}$ *is Lipschitz continuous for some* $\tilde{\omega} \in \mathbb{R}$.

If, in addition, $D(A)$ *is dense in* X, *then* (a) *is equivalent to*
(c) $A \in G(M, \omega, C)$.

Proof (a) $\Rightarrow$ (b). It is clear that for $\lambda > \omega$ and $x \in X$,

$$f(\mu) := \sum_{n=0}^{\infty} (\lambda - \mu)^n (\lambda - A)^{-n-1} Cx$$

defines an analytic function from $\{\mu \in \mathbb{C} : \ |\lambda - \mu| < \lambda - \omega\}$ into X. Since, by Proposition 1.1.5(a), A is closed, we obtain that $R_C(\mu, A)x = f(\mu)$ for $\mu \in \rho_C(A)$. This implies that

$$\frac{d^n}{d\lambda^n} R_C(\lambda, A)x = f^{(n)}(\mu)|_{\mu=\lambda} = (-1)^n n! (\lambda - A)^{-n-1} Cx$$

for $\lambda > \omega$, $x \in X$. Thus, by (1.2.3) and Theorem A.3, there exists $\{S(t)\}_{t \geqslant 0} \subseteq B(X)$ such that $S(0) = 0$, $\{e^{-\omega t} S(t)\}_{t \geqslant 0}$ is Lipschitz continuous, and

$$R_C(\lambda, A)x = \lambda \int_0^{\infty} e^{-\lambda t} S(t)x dt \quad (\lambda > \omega,\ x \in X). \tag{1.2.4}$$

Let $a > \omega$ and $\tilde{C} = R_C(a, A)$. Then $\tilde{C}^{-1}A\tilde{C} = C^{-1}AC = A$ and, by (1.1.4), $(\omega, \infty) \subseteq \rho_{\tilde{C}}(A)$. Define $T(t) \in B(X)$ $(t \geqslant 0)$ by

$$T(t)x = e^{at}\tilde{C}x - S(t)x - a \int_0^t e^{a(t-s)} S(s)x ds \quad (x \in X).$$

Thus $\{e^{-at}T(t)\}_{t \geqslant 0}$ is Lipschitz continuous. Also, by (1.1.4) and (1.2.3), it is not hard to verify that

$$R_{\tilde{C}}(\lambda, A)x = \int_0^{\infty} e^{-\lambda t} T(t) x dt \quad (\lambda > a,\ x \in X),$$

and thus $(A, T(\cdot)) \in G(\tilde{C})$ by Theorem 1.2.1.

(b) $\Rightarrow$ (a). We first note that $C^{-1}AC = \tilde{C}^{-1}A\tilde{C} = A$ and, by (1.1.4) (with $\lambda = a$), $(\tilde{\omega}, \infty) \subseteq \rho_C(A)$. Define $S(t) \in B(X)$ $(t \geqslant 0)$ by

$$S(t)x = R_C(a, A)x - T(t)x - a \int_0^t T(s)x ds \quad (x \in X).$$

Then $S(0)=0$ and $\{e^{-\omega t}S(t)\}_{t\geqslant 0}$ is Lipschitz continuous, where we choose $\omega > \max\{\tilde{\omega}, 0\}$. Also, a simple calculation leads to (1.2.4), and thus by induction,

$$(\lambda - A)^{-n}Cx = \lambda \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_n)}(S(t_1+\cdots+t_n)x - S(t_2+\cdots+t_n)x)dt_1\cdots dt_n$$

for $x \in X$, $\lambda > \omega$ and $n \in \mathbb{N}$. Combining these assertions and noting that $\lambda - \omega < \lambda$, we obtain (1.2.3).

(a) $\Rightarrow$ (c). For $x \in D(A)$ and $\lambda > \max\{\omega, 0\}$, by (1.2.4),

$$\begin{aligned} R_C(\lambda, A)x &= \frac{1}{\lambda}Cx + \frac{1}{\lambda}R_C(\lambda, A)Ax \\ &= \lambda \int_0^\infty e^{-\lambda t}\left(tCx + \int_0^t S(s)Axds\right)dt, \end{aligned}$$

and thus, by the uniqueness of Laplace transforms,

$$S(t)x = tCx + \int_0^t S(s)Axds \quad (t \geqslant 0).$$

Since $\overline{D(A)} = X$ and $\{e^{-\omega t}S(t)\}_{t\geqslant 0}$ is Lipschitz continuous (with Lipschitz constant M), $T(t) := S'(t)$ $(t \geqslant 0)$ is a strongly continuous family in $B(X)$, with $\|T(t)\| \leqslant Me^{\omega t}$ for $t \geqslant 0$. The claim now follows from Theorem 1.2.1.

(c) $\Rightarrow$ (a). This follows easily from Theorem 1.1.3(d) and Proposition 1.1.6. □

From the proof above we see that the constants M, ω in (a) are consistent with that in (c), while the relationship between ω and $\tilde{\omega}$ is as follows: (a) implies that (b) is true for any $\tilde{\omega} > \omega$; (b) implies that (a) is true for any $\omega > \max\{\tilde{\omega}, 0\}$.

We now consider a special class of regularized semigroups.

Definition 1.2.4 A C-regularized semigroup $\{T(t)\}_{t\geqslant 0}$ is *contractive* if $\|T(t)x\| \leqslant \|Cx\|$ for $x \in X$ and $t \geqslant 0$.

Let Y be a subspace of X and let A be a linear operator on X. We denote by A_Y the part of A in Y, i.e., $A_Y \subseteq A$ with maximal domain in Y.

A question arisen immediately from the definition is: If A generates a contraction C-regularized semigroup, does $A_{\overline{R(C)}}$ generates a contraction C_0-semigroup on $\overline{R(C)}$? We will give a counterexample in the next section. Here is a conditional result.

Theorem 1.2.5 *Let $A = C^{-1}AC$, $\overline{C(D(A))} = \overline{R(C)}$ and $D(A) \subseteq R(a - A)$ for some $a > 0$. Then the following are equivalent.*

(a) *A generates a contraction C-regularized semigroup on X.*

(b) *$(0, \infty) \subseteq \rho_C(A)$ and $\lambda\|R_C(\lambda, A)x\| \leqslant \|Cx\|$ for $\lambda > 0$ and $x \in X$.*

(c) *$A_{\overline{R(C)}}$ generates a contraction C_0-semigroup on $\overline{R(C)}$.*