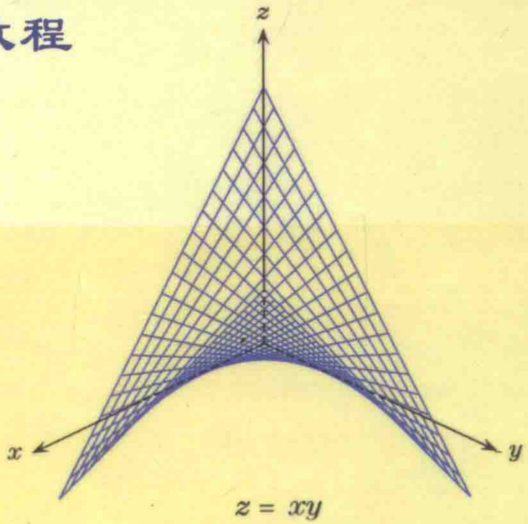
Sudhir R. Ghorpade
Balmohan V. Limaye

UNDERGRADUATE TEXTS IN MATHEMATICS

A Course in Multivariable Calculus and Analysis

多元微积分教程



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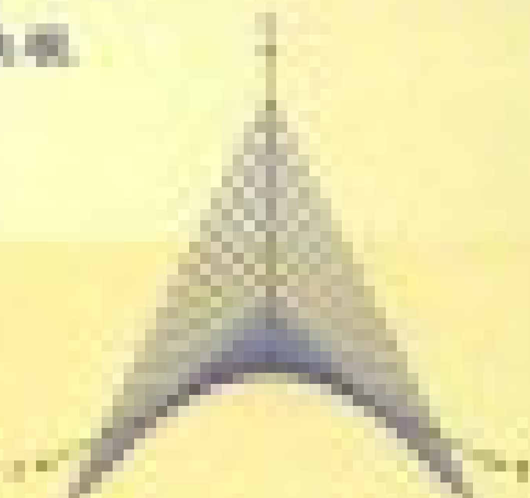

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Preface

Calculus of real-valued functions of several real variables, also known as multivariable calculus, is a rich and fascinating subject. On the one hand, it seeks to extend eminently useful and immensely successful notions in one-variable calculus such as limit, continuity, derivative, and integral to "higher dimensions." On the other hand, the fact that there is much more room to move about in the n-space $\mathbb{R}^n$ than on the real line $\mathbb{R}$ brings to the fore deeper geometric and topological notions that play a significant role in the study of functions of two or more variables.

Courses in multivariable calculus at an undergraduate level and even at an advanced level are often faced with the unenviable task of conveying the multifarious and multifaceted aspects of multivariable calculus to a student in the span of just about a semester or two. Ambitious courses and teachers would try to give some idea of the general Stokes's theorem for differential forms on manifolds as a grand generalization of the fundamental theorem of calculus, and prove the change of variables formula in all its glory. They would also try to do justice to important results such as the implicit function theorem, which really have no counterpart in one-variable calculus. Most courses would require the student to develop a passing acquaintance with the theorems of Green, Gauss, and Stokes, never mind the tricky questions about orientability, simple connectedness, etc. Forgotten somewhere is the initial promise that we shall do unto functions of several variables whatever we did in the previous course to functions of one variable. Also forgotten is a reasonable expectation that new and general concepts introduced in multivariable calculus should be neatly tied up with their relics in one-variable calculus. For example, the area of a bounded region in the plane, defined via double integrals, should be related to formulas for the areas of planar regions between two curves (given by equations in rectangular coordinates or in polar coordinates). Likewise, the volume of a solid in 3-space, defined via triple integrals, should be related to methods for computing volumes of solids of revolution, thereby resolving the mystery that the washer method and the shell method always give the same answer. Indeed, a conscientious student is likely to face a myriad of questions if the promise of extending one-variable calculus to "higher dimensions" is taken seriously. For instance: Why aren't we talking of monotonicity, which was such a big deal in one-variable calculus? Do Rolle's theorem and the mean value theorem, which were considered very important, have genuine analogues? Why is there no L'Hôpital's rule now? Can't we talk of convexity and concavity of functions of several variables, and in that case, shouldn't it have something to do with derivatives? Is it still true that the processes of differentiation and integration are inverses of each other, and if so, then how? Aren't there any numerical methods for approximating double integrals and triple integrals? Whatever happened to infinite series and improper integrals?

We thought and believed that questions and concerns such as those above are perfectly legitimate and should be addressed in a book on multivariable calculus. Thus, about a decade ago, when we taught together a course at IIT Bombay that combined one-variable calculus and multivariable calculus, we looked for books that addressed these questions and could be easily read by undergraduate students. There were a number of excellent books available, most notably, the two volumes of Apostol's *Calculus* and the two-volume *Introduction to Calculus and Analysis* by Courant and John. Besides, a wealth of material was available in classics of older genre such as the books of Bromwich and Hobson. However, we were mildly dissatisfied with some aspect or the other of the various books we consulted. As a first attempt to help our students, we prepared a set of notes, written in a telegraphic style, with detailed explanations given during the lectures. Subsequently, these notes and problem sets were put together into a booklet that has been in private circulation at IIT Bombay since March 1998. Goaded by the positive feedback received from colleagues and students, we decided to convert this booklet into a book. To begin with, we were no less ambitious. We wanted a self-contained and rigorous book of a reasonable size that covered one-variable as well as multivariable calculus, and adequately answered all the concerns expressed above. As years went by, and the size of our manuscript grew, we developed a better appreciation for the fraternity of authors of books, especially of serious books on calculus and real analysis. It was clear that choices had to be made. Along the way, we decided to separate out one-variable calculus and multivariable calculus. Our treatment of the former is contained in *A Course in Calculus and Real Analysis*, hereinafter referred to as ACICARA, published by Springer, New York, in its *Undergraduate Texts in Mathematics* series in 2006.

The present book may be viewed as a sequel to ACICARA, and it caters to theoretical as well as practical aspects of multivariable calculus. The table of contents should give a general idea of the topics covered in this book. It will be seen that we have made certain choices, some quite standard and some rather unusual. As is common with introductory books on multivariable calculus, we have mainly restricted ourselves to functions of two variables. We have also briefly indicated how the theory extends to functions of more than two variables. Wherever it seemed appropriate, we have worked out the generalizations to functions of three variables. Indeed, as explained in the first chapter, there is a striking change as we pass from the one-dimensional world of $\mathbb{R}$ and functions on $\mathbb{R}$ to the two-dimensional space $\mathbb{R}^2$ and functions on $\mathbb{R}^2$. On the other hand, the work needed to extend calculus on $\mathbb{R}^2$ to calculus on the n-dimensional space $\mathbb{R}^n$ for $n > 2$ is often relatively routine. Among the unusual choices that we have made is the noninclusion of line integrals, surface integrals, and the related theorems of Green, Gauss, and Stokes. Of course, we do realize that these topics are very important. However, a thorough treatment of them would have substantially increased the size of the book or diverted us from doing justice to the promise of developing, wherever possible, notions and results analogous to those in one-variable calculus. For readers interested in these important theorems, we have suggested a number of books in the *Notes and Comments* on Chapter 5.

The subject matter of this book is quite classical, and therefore the novelty, if any, lies mainly in the selection of topics and in the overall treatment. With this in view, we list here some of the topics discussed in this book that are normally not covered in texts at this level on multivariable calculus: monotonicity and bimonotonicity of functions of two variables and their relationship with partial differentiation; functions of bounded variation and bounded bivariation; rectangular Rolle's and mean value theorems; higher-order directional derivatives and their use in Taylor's theorem; convexity and its relation with the monotonicity of the gradient and the nonnegative definiteness of the Hessian; an exact analogue of the fundamental theorem of calculus for real-valued functions defined on a rectangle; cubature rules based on products and on triangulation for approximate evaluations of double integrals; conditional and unconditional convergence of double series and of improper double integrals.

Basic guiding principles and the organizational aspects of this book are similar to those in ACICARA. We have always striven for clarity and precision. We continue to distinguish between the intrinsic definition of a geometric notion and its analytic counterpart. A case in point is the notion of a saddle point of a surface, where we adopt a nonstandard definition that seems more geometric and intuitive. Complete proofs of all the results stated in the text, except the change of variables formula, are included, and as a rule, these do not depend on any of the exercises. Each chapter is divided into several sections that are numbered serially in that chapter. A section is often divided into several subsections, which are not numbered, but appear in the table of contents. When a new term is defined, it appears in boldface. Definitions are not numbered, but can be located using the index. Lemmas, propositions, examples, and remarks are numbered serially in each chapter. Moreover, for the convenience of readers, we have often included the statements of certain basic results in one-variable calculus. Each of these appears as a "Fact," and is also serially numbered in each chapter. Each such fact is accompanied by a reference, usually to ACICARA, where a proof can be found. The end of a proof of a lemma or a proposition is marked by the symbol $\square$, while the symbol $\diamond$ marks the end of an example or a remark. Bibliographic details about the books and articles mentioned in the text and in this preface can be found in the list of references. Citations appear in square brackets. Each chapter concludes with *Notes and Comments*, where distinctive features of exposition are highlighted and pointers to relevant literature are provided. These *Notes and Comments* may be collectively viewed as an extended version of the preface, and a reader wishing to get a quick idea of what is new and different in this book might find it useful to browse through them. The exercises are divided into two parts: Part A, consisting of relatively routine problems, and Part B, containing those that are of a theoretical nature or are particularly challenging. Except for the first section of the first chapter, we have avoided using the more abstract vector notation and opted for classical notation involving explicit coordinates. We hope that this will seem more friendly to undergraduate students, while relatively advanced readers will have no difficulty in passing to vector notation and working out analogues of the notions and results in this book in the general setting of $\mathbb{R}^n$.

Although we view this book as a sequel to ACICARA, it should be emphasized that this is an independent book and can be read without having studied ACICARA. The formal prerequisite for reading this book is familiarity with one-variable calculus and occasionally, a nodding acquaintance with 2×2 and 3×3 matrices and their determinants. It would be useful if the reader has some mathematical maturity and an aptitude for mathematical proofs. This book can be used as a textbook for an undergraduate course in multivariable calculus. Parts of the book could be useful for advanced undergraduate and graduate courses in real analysis, or for self-study by students interested in the subject. For teachers and researchers, this may be a useful reference for topics that are skipped or cursorily treated in standard texts.

We thank our parent institution, IIT Bombay, and in particular its Department of Mathematics for providing excellent infrastructure. Financial assistance received from the Curriculum Development Cell at IIT Bombay is gratefully acknowledged. We are indebted to Rafikul Alam, Aldric Brown, Dinesh Karia, Swanand Khare, Rekha Kulkarni, Shobhan Mandal, Thamban Nair, S. H. Patil, P. Shunmugaraj, and especially R. R. Simha, for a critical reading of parts of this book and many useful suggestions. We thank Maria Zeltser for reviewing the entire book. We are especially thankful to Arunkumar Patil, who is mainly responsible for drawing the figures in this book. Thanks are also due to Ann Kostant and her colleagues at Springer, New York, for excellent cooperation, to C. L. Anthony for typing a substantial part of the manuscript, and to David Kramer for his thorough copyediting. Last, but not least, we express our gratitude toward Sharmila Ghorpade and Nirmala Limaye for their continued support. We would appreciate receiving comments, suggestions, and corrections. A dynamic errata together with relevant information about this book will be posted at `http://www.math.iitb.ac.in/~srg/acimc`, and we encourage the reader to visit this website for updates concerning this book.

Mumbai, India *Sudhir Ghorpade*
August 2009 *Balmohan Limaye*

Contents

1

Vectors and Functions

Typically, a first course in calculus comprises of the study of real-valued functions of one real variable, that is, functions $f : D \to \mathbb{R}$, where D is a subset of the set $\mathbb{R}$ of all real numbers. We shall assume that the reader has had a first course in calculus and is familiar with basic properties of real numbers and functions of one real variable. For a ready reference, one may refer to [22], which is abbreviated throughout the text as ACICARA. However, for the convenience of the reader, relevant facts from one-variable calculus will be recalled whenever needed.

The basic object of our study will be the **n-dimensional (Euclidean) space** $\mathbb{R}^n$ consisting of n-tuples of real numbers, namely,

$$\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\},$$

and real-valued functions on subsets of $\mathbb{R}^n$. Whenever we write $\mathbb{R}^n$, it will be tacitly assumed that $n \in \mathbb{N}$, that is, n is a positive integer. Elements of $\mathbb{R}^n$ are sometimes referred to as **vectors** in n-space when $n > 1$. In contrast, the elements of $\mathbb{R}$ are referred to as **scalars**. Given a vector $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and $1 \le i \le n$, the scalar x_i is called the ith **coordinate** of $\mathbf{x}$.

The algebraic operations on $\mathbb{R}$ can be easily extended to $\mathbb{R}^n$ in a componentwise manner. Thus, we define the **sum** of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ to be $\mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_n + y_n)$. It is easily seen that addition defined in this way satisfies properties analogous to those in $\mathbb{R}$. In particular, the **zero vector** $\mathbf{0} := (0, \ldots, 0)$ plays a role similar to the number 0 in $\mathbb{R}$. We might wish to define the **product** of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ to be $(x_1y_1, \ldots, x_ny_n)$. However, this kind of componentwise multiplication is not well behaved. For example, the componentwise product of the nonzero vectors $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$ is the zero vector $(0, 0)$, and consequently, the reciprocals of these nonzero vectors cannot be defined. As a matter of fact, there is no reasonable notion whatsoever of division in $\mathbb{R}^n$, in general. (See the Notes and Comments at the end of this chapter.) Moreover, as explained later, the order relation on $\mathbb{R}$ extends only partially to $\mathbb{R}^n$ when $n > 1$.

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For these reasons, the theory of functions of several variables differs significantly from that of functions of one-variable. However, once $n > 1$, there is not a great deal of difference between the smaller values of n and the larger values of n. This is particularly true with the basic aspects of the theory of functions of several variables that are developed here. With this in view and for the sake of simplicity, we shall almost exclusively restrict ourselves to the case $n = 2$. In this case, the space $\mathbb{R}^n$ can be effectively visualized as the plane. Also, graphs of real-valued functions of two variables may be viewed as surfaces in 3-space. More generally, a surface in 3-space can be given by (the zeros of) a function of three variables. With this in mind, we shall also occasionally allude to $\mathbb{R}^3$ and to real-valued functions of three variables.

In the first section below we discuss a number of preliminary notions concerning vectors in $\mathbb{R}^n$ and some important types of subsets of $\mathbb{R}^n$. Next, in Section 1.2, we develop some basic aspects of (real-valued) functions of two variables. Finally, in Section 1.3, we discuss some useful transformations or coordinate changes of the 3-space $\mathbb{R}^3$.

1.1 Preliminaries

We begin with a discussion of basic facts concerning algebraic operations, order properties, elementary inequalities, important types of subsets, etc. In these matters, there is hardly any simplification possible by restricting to $\mathbb{R}^2$, and thus we will work here with $\mathbb{R}^n$ for arbitrary $n \in \mathbb{N}$.

Algebraic Operations

We have already discussed the notion of addition of points in $\mathbb{R}^n$ and the fact that the corresponding analogue of algebraic properties in $\mathbb{R}$ holds in $\mathbb{R}^n$. More precisely, this means that the following properties hold. Note that each of these is an immediate consequence of the corresponding properties of real numbers. (See, for example, Section 1.1 of ACICARA.)

A1. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ *for all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.
A2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
A3. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.
A4. *Given any* $\mathbf{x} \in \mathbb{R}^n$, *there is* $\mathbf{x}' \in \mathbb{R}^n$ *such that* $\mathbf{x} + \mathbf{x}' = \mathbf{0}$.

These properties may be used tacitly in the sequel. As indicated earlier, we do not have a good notion of multiplication of points in $\mathbb{R}^n$ when $n > 2$. But we have useful notions of *scalar multiplication* and *dot product* that are defined as follows.

Given any $c \in \mathbb{R}$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define

$$c\mathbf{x} := (cx_1, \dots, cx_n).$$

This is referred to as the **scalar multiplication** of the vector $\mathbf{x}$ by the scalar c. Geometrically speaking, the scalar multiple $c\mathbf{x}$ corresponds to stretching or contracting the vector $\mathbf{x}$ according as $c > 1$ or $0 < c < 1$, whereas if $c < 0$, then $c\mathbf{x}$ corresponds to the reflection of $\mathbf{x}$ about the origin followed by stretching or contracting.

Given any $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the **dot product** (also known as the **inner product** or the **scalar product**) of $\mathbf{x}$ and $\mathbf{y}$ is the real number denoted by $\mathbf{x} \cdot \mathbf{y}$ and defined by

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_n y_n.$$

The dot product permits us to talk about the "angle" between two vectors. We shall explain this in greater detail a little later.

We also have an analogue of the notion of the absolute value of a real number, which is defined as follows. Given any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the **norm** (also known as the **magnitude** or the **length**) of $\mathbf{x}$ is the nonnegative real number denoted by $|\mathbf{x}|$ and defined by

$$|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Geometrically speaking, the norm $|\mathbf{x}|$ represents the distance between $\mathbf{x}$ and the origin $\mathbf{0} := (0, \ldots, 0)$. More generally, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the norm of their difference, that is, $|\mathbf{x} - \mathbf{y}|$, represents the distance between $\mathbf{x}$ and $\mathbf{y}$. A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $|\mathbf{u}| = 1$ is called a **unit vector** in $\mathbb{R}^n$. For example, in $\mathbb{R}^2$ the vectors $\mathbf{i} := (1, 0)$ and $\mathbf{j} := (0, 1)$ are unit vectors.

Elementary properties of scalar multiplication, dot product, and the norm are given in the following proposition. It may be remarked that the inequality in (iv) below is a restatement of the **Cauchy–Schwarz inequality** as given in Proposition 1.12 of ACICARA. But the proof given here is somewhat different. The inequality in (v) is referred to as the **triangle inequality**.

Proposition 1.1. *Given any* $r, s \in \mathbb{R}$ *and* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, *we have*

(i) $(rs)\mathbf{x} = r(s\mathbf{x})$, $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ *and* $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$,
(ii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$, $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ *and* $r(\mathbf{x} \cdot \mathbf{y}) = (r\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (r\mathbf{y})$,
(iii) $|\mathbf{x}| \geq 0$; *moreover,* $|\mathbf{x}| = 0 \iff \mathbf{x} = \mathbf{0}$,
(iv) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$,
(v) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$,
(vi) $|r\mathbf{x}| = |r||\mathbf{x}|$.

Proof. Properties listed in (i), (ii), and (iii) are obvious. The inequality in (iv) is obvious if $\mathbf{x} = \mathbf{0}$. Assume that $\mathbf{x} \neq \mathbf{0}$. Let $a := \mathbf{x} \cdot \mathbf{x}$, $b := \mathbf{x} \cdot \mathbf{y}$, and $c := \mathbf{y} \cdot \mathbf{y}$. Then by (iii), $a > 0$. Given any $t \in \mathbb{R}$, consider $q(t) := at^2 + 2bt + c$. In view of (i) and (ii), we have $q(t) = (t\mathbf{x} + \mathbf{y}) \cdot (t\mathbf{x} + \mathbf{y})$, and hence by (iii), $q(t) \geq 0$ for all $t \in \mathbb{R}$. In particular, upon putting $t = -b/a$ and multiplying throughout by a, we obtain $ac - b^2 \geq 0$, that is, $b^2 \leq ac$. Hence $|b| \leq \sqrt{a}\sqrt{c}$, which proves (iv). The inequality in (v) follows from (ii) and (iv), since