Bessel 函数之简单性质*

定义 Bessel 微分方程式

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ 有二特别解 y_1, y_2.

$$y_1=A_1x^n\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2!2^4(n+1)(n+2)}-\cdots\cdots\right],$$

$$y_2=A_2x^{-n}\left[1-\frac{x_2}{2^2(-n+1)}+\frac{x^4}{2!2^4(-n+1)(-n+2)}-\cdots\cdots\right].$$

若令 $A_1=\frac{1}{2^n\pi(n)}, A_2=\frac{1}{2^{-n}\pi(-n)}$, 则得

$y_1=\sum_{r=0}^{\infty}\frac{(-1)^r}{\pi(n+r)\pi(r)}(\frac{x}{2})^{n+2r}$, 以 $J_n(x)$ 表之,

$y_2=\sum_{r=0}^{\infty}\frac{(-1)^r}{\pi(-n+r)\pi(r)}(\frac{x}{2})^{-n+2r}$, 以 $J_{-n}(x)$, 表之

$J_n(x), J_{-n}(x)$ 称为 Bessel 函数.

定理 1 $\frac{d}{dx}(\frac{J_{-n}}{J_n})=-\frac{2\sin n\pi}{\pi x J_n^{\ 2}}$ (J_n, J_{-n} 为 $J_n(x), J_{-n}(x)$ 之简写).

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* 发表于中山大学自然科学, 1930, **11**(2): 76 - 81

由方程式$\frac{d^2 J_n}{dx^2}+\frac{1}{x}\frac{dJ_n}{dx}+(1-\frac{n^2}{x^2})J_n=0$ 及

$$\frac{d^2 J_{-n}}{dx^2}+\frac{1}{x}\frac{dJ_{-n}}{dx}+(1-\frac{n^2}{x^2})J_{-n}=0,$$

消去 J_n 及J_{-n}得

$$(\frac{d^2 J_n}{dx^2}J_{-x}-\frac{d^2 J_{-n}}{dx}J_n)+\frac{1}{x}(\frac{dJ_n}{dx}J_{-n}-\frac{dJ_{-n}}{dx}J_n)=0,$$

$$\therefore \qquad \frac{dJ_n}{dx}J_{-n}-\frac{hJ_{-n}}{dx}J_n=\frac{A}{x}.$$

令两端之 x^{-1}系数相等,易知

$$A=\frac{2n}{\pi(n)\pi(-n)}=\frac{2}{\pi(n-1)\pi(-n)}$$

$$=\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}[1],$$

故

$$\frac{dJ_n}{dx}J_{-n}-\frac{dJ_{-n}}{dx}J_n=\frac{2\sin n\pi}{\pi x},$$

两端以一 J_n^2 除之则得

$$\frac{d}{dx}(\frac{J_{-v}}{J_n})=-\frac{2\sin n\pi}{\pi x J_n^2}.$$

定理 2 若 n 为正整数,则

(1) $J_n(x)=\frac{1}{\pi}\int_0^\pi \cos(n\theta-x\sin\theta)d\theta,$

(2) $J_0(x)=\frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)d\theta,$

(3) $\int_0^\infty e^{-ax}J_0(bx)dx=(a^2+b^2)^{-\frac{1}{2}}.$

证 若 n 为正整数,则

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(n+r)!\,r!}(\frac{x}{2})^{n+2r}.$$

今试观察此级数之公项 $\frac{(-1)^r}{(n+r)!\ r!}(\frac{x}{2})^{n+r+r}$，知可分为 $\frac{1}{(n+r)!}(\frac{x}{2})^{n+r}$ 与 $\frac{(-1)^r}{r!}(\frac{x}{2})^r$ 之乘积，前者为 $e^{\frac{1}{2}xz}$ 中 z^{n+r} 之系数，后者为 $e^{-\frac{1}{2}z^{-1}x}$ 中 z^{-r} 系数，但 $r=0,1,2,\cdots$. 故 $J_n(x)$ 为 $e^{\frac{1}{2}xz}\times e^{-\frac{1}{2}z^{-1}x}=e^{-\frac{1}{2}x(z-\frac{1}{z})}$ 中 z^n 之系数，同样得证 $J_n(x)$ 亦为 $e^{-\frac{1}{2}x(z-\frac{1}{z})}$ 中 $(-1)^n z^{-n}$ 之系数.

由是

$$e^{\frac{1}{z}x}[z-\frac{1}{z}]=J_0+J_1[z-\frac{1}{z}]+J_2[z^2+\frac{1}{z^2}]+\cdots\cdots$$
$$+J_n[z^n+(-1)^n\frac{1}{z^n}]+\cdots\cdots$$

以 $z=e^{i\theta}$ 代入之并应用 Euler 公式

$$\cos(x\sin\theta)=J_0+2J_2\cos2\theta+2J_4\cos4\theta+\cdots\cdots$$
$$\sin(x\sin\theta)=2J_1\sin\theta+2J_3\sin3\theta+\cdots\cdots$$

故

$$\int_0^\pi\cos(x\sin\theta)\cos(2n\theta)\mathrm{d}\theta=\pi J_{2n},$$

$$\int_0^\pi\cos(x\sin\theta)\cos(2n+1)\theta\mathrm{d}\theta=0.$$

$$\int_0^\pi\sin(x\sin\theta)\sin(2n+1)\theta\mathrm{d}\theta=\pi J_{2n+1},$$

$$\int_0^\pi\sin(x\sin\theta)\sin2n\theta\mathrm{d}\theta=0.$$

从此结果知不论 n 为奇为偶，下式恒能成立：

$\int_0^r\cos(n\theta-x\sin\theta)\mathrm{d}\theta=\pi J_n$，此即(1) 式之证.

令 $\theta=\frac{\pi}{2}+\phi\cos(x\cos\phi)=J_0-2J_2\cos2\phi+2J_4\cos4\phi-\cdots\cdots$

$$\therefore J_0 = \frac{1}{\pi}\int_0^{r}\cos(x\cos\phi)\mathrm{d}\phi$$，此即(2)式之证.

$$\therefore \int_0^{\infty} \mathrm{e}^{-ax}J_0(bx)\mathrm{d}x = \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\mathrm{e}^{-ax}\cos(bx\cos\phi)\mathrm{d}x\mathrm{d}\phi$$

$$= \frac{1}{\pi}\int_0^{r}\frac{a\mathrm{d}\phi}{a^2+b^2\cos^2\phi} = (a^2+b^2)^{-\frac{1}{2}}.$$

此即(3)式之证. ▌

定理3　设 $a<1$，即

(1) $$\int_0^{\infty} J_x(ax)J_{-x}(ax)\cos\pi x\mathrm{d}x = \frac{1}{4(1-a^2)^{\frac{1}{2}}}.$$

(2) 若 r 为任意正整数(异于零)，则

$$\int_0^{\infty}\Bigg[J_x(ax)J_{-x}(ax) - \sum_{\substack{r-1\\ m=0}}\frac{(-1)^m(2m)!(\frac{1}{2}ax)^{2m}}{(m!)^2\Gamma(x+m+1)\Gamma(-x+m+1)}\Bigg]\cos\pi x\mathrm{d}x$$

$$=\frac{1}{4}a^{2r}\frac{\Gamma(r+\frac{1}{2})}{\Gamma(r+1)\Gamma(\frac{1}{2})}F(r+\frac{1}{2},1,r+1,a^2).$$[2]

证

$$J_x(ax)J_{-x}(ax)$$

$$=\sum_{\substack{\infty\\ m=0}}\frac{(-1)^m(\frac{1}{2}ax)^m(2m)!}{(m!)^2\Gamma(m+I+x)\Gamma(m+1-x)}$$

$$=\frac{\sin x\pi}{x\pi}\sum_{m=0}^{\infty}\frac{(2m)!(\frac{1}{2}a)^{2m}}{(m!)^2}\frac{x^{2m}}{(x^2-1^2)\cdots(x^2-m^2)}$$

$$\left(因\frac{\sin x\pi}{\pi} = \frac{1}{\Gamma(x)\Gamma(1-x)}\right)$$

$$=\frac{\sin x\pi}{x\pi}\sum_{m=0}^{\infty}\frac{(2m)!\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}+\frac{\sin x\pi}{x\pi}\sum_{m=0}^{\infty}\frac{(2m)!\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}$$

$$\cdot\left\{\frac{x^2 m}{(x^2+1^2)\cdots(x^2-m^2)}-1\right\}.$$

又因 $\displaystyle\int_0^{\infty}\frac{\sin x\pi\cos x\pi}{x}\left\{\frac{x^{2m}}{(x^2-1^2)\cdots(x^2-m^2)}-1\right\}\mathrm{d}x$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\frac{\sin 2x\pi}{x}\times\left[\frac{x^{2m}}{(x^2-1^2)\cdots(x^2-m^2)}-1\right],$$

但依分项分数之理 $\displaystyle\frac{1}{x}\left\{\frac{x^{2m}}{(x^2-1^2)\cdots(x^2-m^2)}-1\right\}=\sum_{r=-m}^{m}\frac{An}{x+r}A_r$

与 x 无关,故两端以 x 乘之,令 $x=\infty$ 求其极限,得

$$\sum_{r=-m}^{m}Ar=0,$$

$$\therefore\qquad\int_{-\infty}^{\infty}\frac{\sin 2x\pi}{x}\left\{\frac{x^{2m}}{(x^2-1^2)\cdots(x^2-m^2)}-1\right\}\mathrm{d}x$$

$$=\sum_{r=-m}^{m}Ar\int_{-\infty}^{\infty}\frac{\sin 2x\pi}{x+r}\mathrm{d}x=\pi\sum_{r=-m}^{m}Ar=0,$$

故 $\displaystyle\int_0^{\infty}J_x(ax)J_{-x}(ax)\cos\pi x\,\mathrm{d}x$

$$=\frac{1}{2\pi}\sum_{m=0}^{\infty}\frac{(2m)!\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}\int_0^{\infty}\frac{\sin 2x\pi}{x}\mathrm{d}x$$

$$=\frac{1}{4}\sum_{m=0}^{\infty}\frac{(2m)!\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}=\frac{1}{4(1-a^2)^{\frac{1}{2}}},$$

$a<1$.

此即(1)式之证.

从上结果易知

$$\int_0^{\infty}\left[J_x(ax)J_{-x}(ax)\right.$$

$$-\sum_{m=0}^{r-1}\frac{(-1)^m(2m)!\left(\frac{1}{2}ax\right)^{2m}}{(m!)^2\Gamma(x+m+1)\Gamma(-x+m+1)}\Bigg]\cos\pi x\,\mathrm{d}x$$

$$=\int_0^\infty\frac{\sin 2x\pi}{2\pi x}\mathrm{d}x\sum_{m=r}^{\infty}\frac{(2m!)\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}$$

$$=\frac{1}{4}\sum_{m=r}^{\infty}\frac{(2m)!\left(\frac{1}{2}a\right)^{2m}}{(m!)^2}$$

$$=\frac{1}{4}\frac{\Gamma\left(r+\frac{1}{2}\right)a^{2r\,*}}{\Gamma(r+1)\Gamma\left(\frac{1}{2}\right)}$$

$$\cdot\left[1+\frac{r+\frac{1}{2}}{(r+1).1}a^2+\frac{\left(r+\frac{1}{2}\right)\left(r+1+\frac{1}{2}\right)1.2}{1.2(r+1)(r+2)}a^4\right.$$

$$=\left.\frac{\left(r+\frac{1}{2}\right)\left(r+1+\frac{1}{2}\right)\left(r+2+\frac{1}{2}\right)1.2.3}{1.2.3.(r+1)(r+2)(r+3)}a^6+\cdots\right]$$

$$=\frac{1}{4}\frac{\Gamma\left(r+\frac{1}{2}\right)a^{2r}}{\Gamma(r+1)\Gamma\left(\frac{1}{2}\right)}\cdot F\left(r+\frac{1}{2},1,r+1,a^2\right).$$

当 $a<1$ 上式之超越几何级数为收敛[4].

此篇之成,吾师何衍璿先生,曾校阅过,多为指正,谨此致谢.

$$*\quad\frac{(2r)!\left(\frac{1}{2}a\right)^{2r}}{(r!^2)}=\frac{r\left(r-\frac{1}{2}\right)(r-1)\left(r-\frac{1}{2}\right)(r-2)\left(r-\frac{5}{2}\right)\cdots\frac{1}{2}}{(r!)^2}a^{2r}.$$

$$=\frac{r(r-1)\cdots1.\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right.}{(r!)^2\Gamma\left(\frac{1}{2}\right)}a^{2r}=\frac{\Gamma\left(r+\frac{1}{2}\right)a^{2r}}{\Gamma(r+1)\Gamma\left(\frac{1}{2}\right)}.$$

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[1] Williamson Integral caleulus, p162.

[2] Part Ⅱ Vol XXV Cambridge Philosophical Society, Fox: A note on some Ingegrals Involving Bessel Functions.

[3] Forzyth a. Treatise on Differential equations.

[4] Whitlaker and Watson Modern Analysis, p24.

行列式之一性质*

余尝读解析几何学补锥面之理论，忽推出行列式之一性质，理虽浅显，然亦足见几何性质恒有功于代数运算也，故录之于此.

定理 由二次锥面(C)之顶点作直线垂直于(C)之切面，则此直线之轨迹亦为二次锥面(C')，以(C)之补锥面名之，反言之，(C')之补锥面为(C).

设正交位标轴之顶点为原点，令(C)之方程为

$$Ax^2 + A'y^2 + A''z^2 = 0. \tag{1}$$

又设(C)之元线之方向参数为α, β, γ，则

$$A\alpha^2 + A'\beta^2 + A''\gamma^2 = 0. \tag{2}$$

过原点而垂直于(C)之切面之直线为

$$\frac{x}{A\alpha} = \frac{y}{A'\beta} = \frac{z}{A''\gamma}.$$

由(1)，(2)消去α, β, γ得方程式如次

$$\frac{x^2}{A} + \frac{y^2}{A'} + \frac{z^2}{A''} = 0.$$

此为二次锥面(C')，同理可推出(C')之补锥面之方程式

$$\frac{x^2}{\frac{1}{A}} + \frac{y^2}{\frac{1}{A'}} + \frac{z^2}{\frac{1}{A''}} = 0.$$

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即(C)之方程式也,故得定理之证.

问题 设位标轴为正交,二次锥面(C)之方程式为

$$\varphi(x,y,z)=Ax^2+A'y^2+A''z^2+2Byz+2B'zx+2B''xy=0.$$

试求此锥面之补锥面(C').

(C') 之方程式可由下列二式消去 α,β,γ 而得.

(1) $\varphi(\alpha,\beta,\gamma)=0$.

(2) $\frac{x}{\varphi'\alpha}=\frac{y}{\varphi'\beta}=\frac{z}{\varphi'\gamma}$.

(2)式又可以下式代之

$$\begin{cases}\frac{1}{2}\varphi\alpha'-\lambda x=0\\ \frac{1}{2}\varphi\beta'-\lambda y=0\\ \frac{1}{2}\varphi\alpha'-\lambda z=0\end{cases}\tag{3}$$

以 α,β,γ 依次乘(3)中各式之两端而求其和,则得

$$\varphi(\alpha,\beta,\gamma)-\lambda(\alpha x+\beta y+\gamma z)=0.$$

计及(1)式,即有

$$\alpha z+\beta y+\gamma z=0.\tag{4}$$

由(3),(4)消去 α,β,γ 得(C')之方程式如次

$$\begin{vmatrix}A & B'' & B' & x\\ B'' & A' & B & y\\ B' & B & A'' & z\\ x & y & z & 0\end{vmatrix}=0.$$

即 $$ax^2+a'y^2+a''z^2+2byz+2b'zx+2b''xy=0.\tag{5}$$

就中

$$a=\begin{vmatrix}A' & B\\ B & B''\end{vmatrix},\quad a'=\begin{vmatrix}A & B'\\ B' & A''\end{vmatrix},\quad a''=\begin{vmatrix}A & B''\\ B'' & A'\end{vmatrix};$$

$$b=\begin{vmatrix} B' & A \\ B & B'' \end{vmatrix},\quad b'=\begin{vmatrix} B'' & A' \\ B' & B \end{vmatrix},\quad b''=\begin{vmatrix} B & B'' \\ A'' & B' \end{vmatrix}.$$

结论:由上述定理可知(C')之补锥面应为(C),故由(5)式依上法求之,则(C)之方程式之形为

$$k(Ax^2+A'y^2+A''z^2+2Byz+2B'zx+2B''xy)=0.$$

k 为常数,故有下述结果

$$\frac{\begin{vmatrix} a' & b \\ b & a'' \end{vmatrix}}{A}=\frac{\begin{vmatrix} a & b' \\ b' & a'' \end{vmatrix}}{A'}=\frac{\begin{vmatrix} a & b'' \\ b'' & a' \end{vmatrix}}{A''}$$

$$=\frac{\begin{vmatrix} b' & a \\ b & b'' \end{vmatrix}}{B}=\frac{\begin{vmatrix} b'' & a' \\ b' & b \end{vmatrix}}{B'}=\frac{\begin{vmatrix} b & b'' \\ a'' & b' \end{vmatrix}}{B''}=\frac{\Delta_1}{\Delta}=\Delta.$$

就中

$$\Delta=\begin{vmatrix} A & B'' & B' \\ B'' & A' & B \\ B' & B & A'' \end{vmatrix},\quad \Delta_1=\begin{vmatrix} a & b'' & b' \\ b'' & a' & b \\ b' & b & a'' \end{vmatrix}.$$

故

$$a'a''-b^2=A\Delta,$$
$$aa''-b'^2=A'\Delta,$$
$$aa'-b''^2=A''\Delta,$$
$$b'b''-ab=B\Delta,$$
$$b''b-a'b'=B'\Delta,$$
$$bb'-a''b''=B''\Delta.$$

On Meromorphic Functions of Infinite Order Ⅰ*

The present paper is based upon the works of Blumenthal and R. Nevanlinna.

Considering the characteristic function, we can construct the net orders of every meromorphic function of infinite order, by means of the method of Blumenthal.

The second fundamental theorem of R. Nevanlinna is simplified in the case of meromorphic functions of infinite order. Following the new form of that theorem, the author has generalized the theorem of Picard-Borel.

The author is indebted much to Prof. Takenouchi, and Prof. Tsuji, for their valuable criticisms.

Ⅰ The Orders of Meromorphic Functions of Infinite Order

1. Given a meromorphic function $f(x)$ with its characteristic function

$$T(r,f) = m(r,\infty) + N(r,\infty),$$

we define that the function $f(x)$ is of infinite order, if

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$$\lim_{r \to \infty} \frac{\log T(r, f)}{\log r} = \infty .$$

Since the characteristic function $T(r, f)$ is a continuous function of r it can be written in the form

$$T(r, f) = r^{\lambda(r)},$$

where $\lambda(r)$ is also a continuous function of r , and

$$\lim_{r \to \infty} \lambda(r) = \infty .$$

Blumenthal[1] proved that there exists an ensemble of asymptotic function types which adjoin to the function $\lambda(r)$ and to a suitably chosen infinitesimal[2]. We call a positive decreasing continuous function of x an *infinitesimal, if it tends toward zero with* $\frac{1}{x}$.

Every function $\mu(r)$ in the ensemble satisfies the following conditions:

1° $\mu(r)$ is a positive non-decreasing continuous function of r and

$$\lim_{r \to \infty} \mu(r) = \infty,$$

2° $\mu\left(r^{1+\frac{1}{\mu(r)\eta}}\right) \leqslant \mu(r)^{1+\eta}$ for all values of r .

3° $\mu(r) < \lambda(r)^{1+\delta(r)}$ for a suitably chosen infinitesimal $\delta(r)$ and at least for a sequence of r which tends toward infinity,

4° $\mu(r) \geqslant \lambda(r)$ for all sufficiently great values of r .

5° $\mu(r)^{\eta(r)}$ tends toward infinity with r .

We define that the functions $\mu(r)$ are the net orders, and that the function $\lambda(r)$ is the brut order, of the meromorphic function $f(x)$.

2. Since $\lambda(r) \leqslant \mu(r)$ for all sufficiently great values of r we get

$$T(r, f) \leqslant r^{\mu(r)} .$$

By means of the formula

$$m(r, z) + N(r, z) = T(r, f) + h(r, z),$$ [3]

$h(r, z)$ being bounded, we conclude that

$$N(r, z) < r^{\mu(r)^{1+\delta(r)}},$$

where $\delta(r)$ is an infinitesimal.

Arranging all the n_1 roots

$$a_1, a_2, a_3, \cdots, a_n \cdots$$

of the equation

$$f(x) = z. \quad (1)$$

in the closed region $|x| \leqslant r$, in the order of their absolute values

$$r_1, r_2, r_3, \cdots, r_n, \cdots$$

and supposing that the origin is not a root of the equation (1), we get:

$$\begin{aligned} N(r, z) &= \sum_{p=1}^{n_1} \log \frac{r}{r_p} \\ &= \log \frac{r^{n_1}}{r_1 \cdot r_2 \cdots r_{n_1}} \\ &= \log \frac{r^n}{r_1 \cdot r_2 \cdots r_n} \cdot \frac{r^{n_1 - n}}{r_{n+1} \cdots r_{n_1}} \\ &\geqslant \log \frac{r^n}{r_1 \cdot r_2 \cdots r_n} \\ &\geqslant \log \left(\frac{r}{r_n}\right)^n. \end{aligned}$$

We conclude consequently, that the number n of roots in the circle $|x| = r$, whose absolute values $\leqslant r_n$, satisfies the following inequality:

$$n < \frac{r^{\mu(r)^{1+\delta(r)}}}{\log r - \log r_n}.$$

Let

$$r = r_n \exp \frac{1}{r_n^{\mu(r_n)}},$$

then

$$n < r_n^{\mu(r_n)} r^{\mu(r)^{1+\delta(r)}} < 2^{2\mu(r)^{1+\delta(r)}}.$$

An arbitrary infinitesimal $\varepsilon(r)$ satisfies the inequality:

$$r_n \exp \frac{1}{r_n^{\mu(r_n)}} < r_n^{1+\frac{1}{\mu(r_n)^{\varepsilon(r_n)}}},$$

that is

$$r < r_n^{1+\frac{1}{\mu(r_n)^{\varepsilon(r_n)}}},$$

for all sufficiently great values of r_n .

If we take $\varepsilon(r)$ as the adjoined infinitesimal $\eta(r)$ of $\mu(r)$, then

$$\mu(r) \leqslant \mu(r_n)^{1+\eta(r_n)}.$$

Hence

$$n < r_n^{2(1+\frac{1}{\mu(r_n)^{\eta(r_n)}})} \mu(r_n)^{(1+\eta(r_n))(1+\delta(r_n))} < r_n^{4\mu(r_n)^{1+\eta_1(r_n)}},$$ [4]

for all sufficiently great values of r_n .

Choosing an infinitesimal $\eta'(r)$ such that $\eta'(r) \geqslant \eta_1(r)$ and that $\mu(r)^{\eta'(r)}$ increase (at least does not decrease), we conclude:

$$n < r_n^{4\mu(r_n)^{1+\eta'(r_n)}} = r_n^{\mu(r_n)^{1+\delta_1(r_n)}}.$$

Hence

$$r_n^{\mu(r_n)^{1+\delta_1(r_n)}} \leqslant r^{\mu(r)^{1+\delta_1(r)}},$$

for all values of $r > r_n$.

Consequently, the following theorem is proved.

Theorem I The number $n(r, z)$ of the roots of the equation (1) in the closed region $|x| \leqslant r$ satisfies the following inequality, for all sufficiently great values of r :

$$n(r, z) < r^{\mu(r)^{1+\delta_1(r)}}. \tag{2}$$

3. In this article we are going to prove the following theorem:

Theorem II For a meromorphic function $f(x)$ of net order $\mu(r)$ and for every infinitesimal $\delta'(r)$ the inequality

$$T(r, f) < \mu(r)^{1+\delta'(r)}\log r$$

does not hold good for all sufficiently great values of r.

Writing $r^{\lambda(r)}$ for $T(r, f)$, we are going to prove that the reduced form

$$\lambda(r)\log\gamma < (1 + \delta'(r))\log\mu(r) + \log_2 r$$

does not hold good for all sufficiently great values of r.

If

$$\lambda(r)\log r < (1 + \delta'(r))\log\mu(r) + \log_2 r$$

for all sufficiently great values of r, then

$$\lambda(r)\log r < \log r(\log\mu(r) + 1),$$

whence follows

$$\lambda(r) < \log\mu(r) + 1.$$

Since $\mu(r) < \lambda(r)^{1+\delta(r)}$ for the suitably chosen infinitesimal $\delta(r)$ and for at least a sequence $E(r)$ of r which tends toward infinity, we get:

$$\log\mu(r) < (1 + \delta(r))\log\lambda(r)$$

for every values of r in $E(r)$.

The function $\mu(r)$ increases toward infinity with r, hence $\lambda(r)$ is not bounded when r situates in $E(r)$.

Therefore the function $\lambda(r)$ increases toward infinity, when r varies, in a partial sequence $E_1(r)$ of r, toward infinity.

Consequently, when r varies toward infinity in the sequence

$E_1(r)$ we get:

$$\operatorname{Lim} \frac{\log\lambda(r)}{\lambda(r)} = 0.$$

Therefore when r situates in $E_1(r)$ and is sufficiently great we have

$$\log\lambda(r) < \varepsilon\lambda(r)$$

where ε is an arbitrary positive constant however small. Hence for sufficiently great values of r in $E_1(r)$ we have:

$$\log\mu(r) < (1+\delta)\varepsilon\lambda(r).$$

Hence

$$\log\mu(r) + 1 < (1+\delta)\varepsilon\lambda(r) + 1.$$

Choosing $\varepsilon < \frac{1}{4}$, and r from $E_1(r)$ so great that

$$1 + \delta(r) < 2,\ \log\lambda(r) < \varepsilon\lambda(r),\ \frac{\lambda(r)}{2} > 1,$$

we get:

$$\log\mu(r) + 1 < \lambda(r)$$

for sufficiently great values of r in $E_1(r)$.

This contradiction proves the theorem.

Ⅱ The Theorem of Picard-Borel

1. R. Nevanlinna proved the following result:

Every meromorphic function $f(x)$ and its derivative $f'(x)$ satisfy the inequality:

$$m\left(r, \frac{f'}{f}\right) < 24 + 3 \overset{+}{\log} \left|\frac{1}{c_0}\right| + 2 \overset{+}{\log} \frac{1}{r} + 4\log^{+} \rho$$

$$+ 3 \overset{+}{\log} \frac{1}{\rho - r} + 4 \overset{+}{\log}\, T(\rho, f) \qquad (3)$$

where $0 < r < \rho$, supposing $f(0) = c_0 \neq 0, \infty$[5].

Now, we are going to simplify this inequality.

Suppose that $\mu(r)$ is a net order of $f(x)$ and that it adjoins to an infinitesimal $\eta(r)$.

Let $\rho = r^{1+\frac{1}{\mu(r)^{\eta(r)}}}$, then

$$T(\rho, f) \leqslant \rho^{\mu(\rho)} \leqslant \left(r^{1+\frac{1}{\mu(r)^{\eta(r)}}}\right)^{\mu(r)^{1+\eta(r)}}.$$

Since $\mu(r)^{\eta(r)}$ tends toward infinity we get

$$T(\rho, f) < r^{2\mu(r)^{1+\eta(r)}}$$

for all sufficiently great values of r.

Therefore, when r is sufficiently great,

$$\overset{+}{\log}\, T(\rho, f) = \log T(\rho, f) < 2\mu(r)^{1+\eta(r)} \log r,$$

$$\overset{+}{\log}\, \rho = \log \rho = \left(1 + \frac{1}{\mu(r)^{\eta(r)}}\right) \log r < 2 \log r,$$

$$\frac{1}{\rho - r} = \frac{1}{r\left(r^{\frac{1}{\mu(r)^{\eta(r)}}} - 1\right)}.$$

But

$$r^{\frac{1}{\mu(r)^{\eta(r)}}} = \exp \frac{1}{\mu(r)^{\eta(r)}} \log r = 1 + \frac{\log r}{\mu(r)^{\eta(r)}} + \cdots,$$

hence

$$r^{\frac{1}{\mu(r)^{\eta(r)}}} - 1 > \frac{\log r}{\mu(r)^{\eta(r)}}.$$

Therefore

$$\frac{1}{\rho - r} < \frac{\mu(r)^{\eta(r)}}{r \log r},$$

$$\overset{+}{\log} \frac{1}{\rho - r} < \overset{+}{\log} \frac{\mu(r)^{\eta(r)}}{r \log r} < \mu(r)^{\eta(r)},$$

where r is sufficiently great.