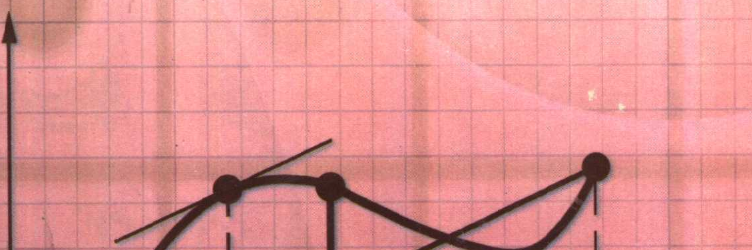
[大学数学]

College Mathematics

王冬冬　郭　云　编著



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（大学数学）

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CHAPTER 1 FUNCTIONS

In this chapter we will review the basic properties of real numbers, introduce the concept of function, and discuss different types of functions.

§ 1 SETS AND REAL NUMBERS

Sets will usually be denoted by capital letters:

$$A, B, C, \cdots, X, Y, Z,$$

and elements by lower-case letters: $a, b, c, \cdots, x, y, z$. We write $x \in S$ to mean "x is an element of S", or "x belongs to S". If x does not belong to S, we write $x \notin S$. We sometimes designate sets by displaying the elements in braces; for example, the set of positive even integers less than 10 is denoted by $\{2, 4, 6, 8\}$. If S is the collection of all x which satisfy a property P, then we indicate this briefly by writing $S = \{x \mid x \text{ satisfies } P\}$.

From a given set we can form new sets, called **subsets** of the given set. For example, the set consisting of all positive integers less than 10 which are divisible by 4, namely, $\{4, 8\}$, is a subset of the set of even integers less than 10. In general, we say that a set A is a subset of B, and we write $A \subseteq B$ whenever every element of A also belongs to B. The statement $A \subseteq B$ does not rule out the possibility that $B \subseteq A$. In fact, we have both $A \subseteq B$ and $B \subseteq A$ if, and only if, A and B have the same elements. In this case we shall call the sets A and B equal and we write $A = B$. If A and B are not equal, we write $A \neq B$. If $A \subseteq B$ but $A \neq B$, then we say that A is a proper subset of B.

It is convenient to consider the possibility of a set which contains no elements whatever; this set is called the **empty set** and we agree to call it a subset of every set. You may find it helpful to picture a set as a box containing certain objects, its elements. The empty set is then an empty box. We denote the empty set by the symbol $\varnothing$.

Certain sets of real numbers, called intervals, appear with great frequency in calculus. They can be grouped into nine categories.

Name	Notation	Description
Open interval	(a,b)	$\{x \mid a < x < b\}$
Closed interval	$[a,b]$	$\{x \mid a \leqslant x \leqslant b\}$
Half-open interval	$[a,b)$	$\{x \mid a \leqslant x < b\}$
Half-open interval	$(a,b]$	$\{x \mid a < x \leqslant b\}$
Infinite interval	$(a,+\infty)$	$\{x \mid x > a\}$
Infinite interval	$(-\infty,a)$	$\{x \mid x < a\}$
Infinite interval	$[a,+\infty)$	$\{x \mid x \geqslant a\}$
Infinite interval	$(-\infty,a]$	$\{x \mid x \leqslant a\}$
Infinite interval	$(-\infty,+\infty)$	$\{x \mid -\infty < x < +\infty\}$

Intervals of the form (a,b), $[a,b]$, $[a,b)$, and $(a,b]$ are bounded intervals, and intervals of the form $(a,+\infty)$, $(-\infty,a)$, $[a,+\infty)$, $(-\infty,a]$ and $(-\infty,+\infty)$ are unbounded intervals.

§ 2 FUNCTIONS

We often consider correspondences between two sets of numbers. For example, each temperature in degrees Celsius corresponds to a temperature in degrees Fahrenheit. There is also a correspondence between the radius of a circle and the circle's area. In mathematics such correspondences are called functions

A **function** consists of a domain and a rule. The **domain** is a set of real numbers, whose elements are usually denoted by lower case letters. $a,b,c,\cdots,z$. The **rule** assigns to each number in the domain one and only one number.

Functions are normally denoted by f, g, or h. The value assigned by a function f to a member x of its domain is written $f(x)$ and is read "f of x" or "the value of f at x." The collection of values (numbers) $f(x)$ that a given function assigns to the members of its domain is called the **range** of f.

We can think of a function as a machine that takes the members of the domain and applies the rule to each to produce the members of the range. Any member x of the domain goes into a member $f(x)$ of the range.

We stress two key points in the definition: First, a function must make an assignment to each number in the domain. For example, if the domain of a function f is $(-\infty,+\infty)$, then f must make an assignment to each real number. Second, a function can assign only one number to any given number in the domain. Thus f cannot assign both 5 and 6 to a single number in its domain.

We give several examples of functions. Let f be the function whose domain consists of all real numbers and whose rule assigns to any real x the number x^2-1. Then we write

$$f(x)=x^2-1 \quad \text{for all} \quad x.$$

Next, if the domain of a function g consists of all real numbers except 3 and if g assigns

$$\frac{x-1}{x-3}$$

to each such number, then g is described by

$$g(x)=\frac{x-1}{x-3} \quad \text{for} \quad x\neq 3.$$

Finally, the function h whose domain consists of all number greater than or equal to -273.15 and assigns to each number x the number $\frac{9}{5}x+32$ can be written

$$h(x)=\frac{9}{5}x+32 \quad \text{for} \quad x\geqslant -273.15.$$

When the rule of a function is described by one formula or equation, we normally specify the numbers in the domain after the rule. If the domain consists of all real numbers for which the formula or equation is meaningful, then formula or equation is meaningful, then we may omit mention of the domain. Thus we may write

$$f(x)=x^2-1 \quad \text{and} \quad g(x)=\frac{x-1}{x-3}.$$

Without specifying the rule for f and g, but it is necessary to give the domain of h if

$$h(x)=\frac{9}{5}x+32 \quad \text{for} \quad x\geqslant-273.15.$$

This is true because the rule for h is meaningful for numbers less than -273.15, although these numbers are explicitly excluded from the domain.

When there can be no misinterpretation, we will sometimes let the rule of a function stand for the function itself. For instance, we can replace f and g defined above by the rules x^2-1 and $(x-1)(x-3)$, respectively.

Example 1 Describe the function f that associates with each temperature in degrees Celsius the corresponding temperature in degrees Fahrenheit.

Solution Let x be the temperature in degrees Celsius. Then the temperature in degrees Fahrenheit is $f(x)$, we find it by applying the formula

$$f(x)=\frac{9}{5}x+32.$$

We restrict x to values not below absolute zero, which for practical purposes is -273.15, so the domain of f is $[-273.15,+\infty)$. Therefore the function f is given by

$$f(x)=\frac{9}{5}x+32 \quad \text{for} \quad x\geqslant -273.15.$$

Example 2 Substance A undergoes a chemical reaction to become substance B. The amount of A present initially is 3 grams. The rate $f(x)$ at which x grams of A are turned into B is proportional to the product of x and $3-x$. Express $f(x)$ in terms of x.

Solution Since $f(x)$ is proportional to x and $3-x$, there is a number $c\neq 0$ such that

$$f(x)=cx(3-x).$$

There are initially 3 grams of substance A, so this relation holds only for $0\leqslant x\leqslant 3$. Consequently

$$f(x)=cx(3-x) \quad \text{for} \quad 0\leqslant x\leqslant 3.$$

Example 3 $f(x)=2$ for all x.

Example 4 $f(x)=|x|=\begin{cases} x & x\geqslant 0 \\ -x & x<0 \end{cases}$ for all x.

§ 3 SOME BASIC PROPERTIES OF FUNCTIONS

1. Increasing and decreasing functions

A function f is said to be **increasing** on an interval I

provided that $f(x) < f(y)$ whenever x and y are in I and $x < y$; A function f is said to be **decreasing** on an interval I provided that $f(x) < f(y)$ whenever x and y are in I and $x > y$.

Graphically, a function is increasing on an interval I if its graph slopes upward to the right. It is decreasing on I if its graph slopes downward to the right. For example if $f(x) = x^2$, then f is decreasing on $(-\infty, 0)$ and increasing on $[0, +\infty)$. If $g(x) = x^3$ then g is increasing on $(-\infty, +\infty)$.

2. Even and odd functions

A function is called **even function** if $-x$ is in the domain whenever x is and $f(-x) = f(x)$ for all x in the domain; a function is called **odd function** if $-x$ is in the domain whenever x is and $f(-x) = -f(x)$ for all x in the domain. Functions such as $y = x^2$, $y = 5x^6 - 3x^2 + 4$ and $y = \cos 3x$ are even functions; functions such as $y = x^3$, $y = 5x^3 - 3x$ and $y = \sin 3x$ are odd functions. Even functions have graphs that are symmetric with respect to the y axis. Odd functions have graphs that are symmetric with respect to the point $(0, 0)$. By using the knowledge of symmetry properties we can reduces the work of sketching graphs.

3. Bounded and unbounded functions

Let function $f(x)$ be determined in the interval I. If there is a positive number M such that

$$|f(x)| \leqslant M \quad \text{for all } x \in I$$

then we say $f(x)$ is a **bounded function** on the interval I. Otherwise we call it unbounded function. Functions such as $y=\sin x$ and $y=\frac{x}{1+x^2}$, are bounded ones in $(-\infty, +\infty)$ while $y=\frac{1}{x}$ and $y=e^x$ are not in $(0, +\infty)$.

4. Periodic functions

A function with the property

$$f(x)=f(x+T) \quad \text{for all} \quad x \in \mathbf{R}$$

where $T>0$, is called **periodic function**, with period T. Functions such as $y=\sin x$ and $y=\tan x$ are periodic function with period 2π and π respectively.

§ 4 INVERSE FUNCTIONS

1. Definition

Let f be a function. Then f has an **inverse** provided that there is a function such that the domain of g is the range of f and such that

$$f(x)=y \quad \text{if and only if} \quad g(y)=x$$

for all x in the domain of f and all y in the range of f. The function g, which is uniquely determined, is also denoted by f^{-1}. Since we usually use x as the independent variable, we replace y by x and obtain

$$g(x)=f^{-1}(x).$$

2. Properties of inverses

First, observe that the domain of f^{-1} is the range of f, while the range of f^{-1} is the domain of f. From this observation we can derive three elementary properties of inverses, which we group together in a theorem.

Theorem 1 Let f have an inverse. Then the following statements hold.

(1) The function f^{-1} has an inverse, and

$$(f^{-1})^{-1}=f.$$

(2) For all in the domain of we have

$$f^{-1}(f(x))=x.$$

(3) For all y in the range of f we have

$$f(f^{-1}(y))=y.$$

(4) A function f has an inverse if and only if for any two numbers x_1 and x_2 in the domain of f,

$$\text{if } x_1\neq x_2,\ \text{then } f(x_1)\neq f(x_2).$$

(5) Every increasing function (and every decreasing function) has an inverse.

Proof We only give the proof of (5).

Let f be increasing, and suppose that x_1 and x_2 are in the domain of f and $x_1\neq x_2$. Then either $x_1<x_2$, in which case $f(x_1)<f(x_2)$, or $x_1>x_2$, in which case $f(x_1)>$

$f(x_2)$. In either case, $f(x_1) \neq f(x_2)$. Therefore f satisfies the criterion in (4) and hence has an inverse. The proof for decreasing functions is similar.

The clearest way to describe an inverse of a function is often to write its rule by one or more formulas.

(1) Write $y=f(x)$.

(2) Solve for x in terms of y.

(3) Change x for y.

Let us see how the method works.

Example Let $f(x)=3x-2$. Write a formula for the inverse of f.

Solution Following the steps listed. We obtain

$$y=3x-2$$

$$x=\frac{y+2}{3}$$

so that

$$f^{-1}(x)=\frac{x+2}{3}.$$

3. Graphs of inverses

To find a method of graphing the inverse of f, we begin by noting that if $(-1, 4)$ is on the graph of f, then $f(-1)=4$, so that $f^{-1}(4)=-1$. This means that $(4, -1)$ is on the graph of f^{-1}. But $(-1,4)$ and $(4,-1)$ are symmetric with respect to the line $y=x$, as are (a,b) and (b,a). Thus the graph of f^{-1} is obtained by f simply reflecting the graph of f through the line $y=x$.

§ 5 COMPOSITE FUNCTIONS

If f and g are any two functions, we define a new function $f \circ g$, the composition of f and g, by

$$(f \circ g)(x) = f(g(x))$$

the domain of $f \circ g$ is $\{x \mid x$ is in domain g and $g(x)$ is in domain $f\}$. The expression $f \circ g$ is read "f circle g". The function $f \circ g$ is the result of performing g and then performing f.

Example 1 Determine the function $f(g(x))$ and $g(f(x))$, where

$$f(x) = \sqrt{2x-3} \quad \text{and} \quad g(x) = \frac{1}{x}$$

find $g(f(4))$ and $f\left(g\left(\frac{1}{6}\right)\right)$.

Solution $f(g(x)) = f\left(\frac{1}{x}\right) = \sqrt{\frac{2}{x} - 3} \quad (0 < x \leqslant \frac{2}{3})$

$$g(f(x)) = g(\sqrt{2x-3}) = \frac{1}{\sqrt{2x-3}} \quad (x > \frac{3}{2})$$

$$g(f(4)) = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$f\left(g\left(\frac{1}{6}\right)\right) = \sqrt{9} = 3.$$

Example 2 Determine the function $f(g(x))$ and $g(f(x))$, where

$$f(x) = \arccos x \quad \text{and} \quad g(x) = \lg x.$$