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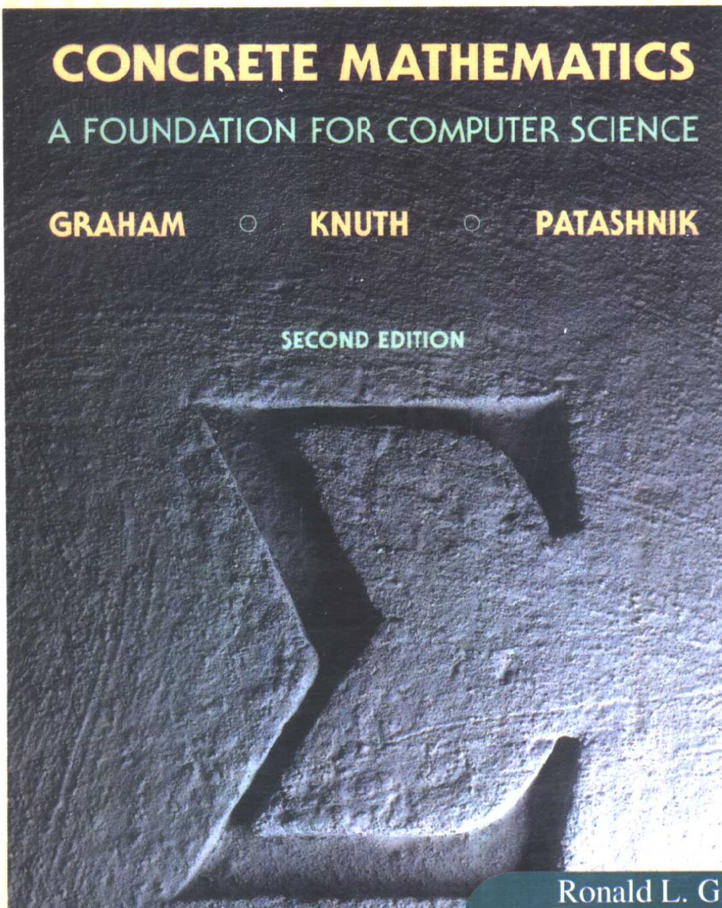
具体数学

计算机科学基础

（英文版·第2版）



Ronald L. Graham
（美） Donald E. Knuth 著
Oren Patashnik

机械工业出版社
China Machine Press

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Concrete Mathematics

A Foundation for Computer Science

(Second Edition)

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出版者的话

文艺复兴以降，源远流长的科学精神和逐步形成的学术规范，使西方国家在自然科学的各个领域取得了垄断性的优势；也正是这样的传统，使美国在信息技术发展的六十多年间名家辈出、独领风骚。在商业化的进程中，美国的产业界与教育界越来越紧密地结合，计算机学科中的许多泰山北斗同时身处科研和教学的最前线，由此而产生的经典科学著作，不仅擘划了研究的范畴，还揭橥了学术的源变，既遵循学术规范，又自有学者个性，其价值并不会因年月的流逝而减退。

近年，在全球信息化大潮的推动下，我国的计算机产业发展迅猛，对专业人才的需求日益迫切。这对计算机教育界和出版界都既是机遇，也是挑战；而专业教材的建设在教育战略上显得举足轻重。在我国信息技术发展时间较短、从业人员较少的现状下，美国等发达国家在其计算机科学发展的几十年间积淀的经典教材仍有许多值得借鉴之处。因此，引进一批国外优秀计算机教材将对我国计算机教育事业的发展起积极的推动作用，也是与世界接轨、建设真正的世界一流大学的必由之路。

机械工业出版社华章图文信息有限公司较早意识到“出版要为教育服务”。自1998年始，华章公司就将工作重点放在了遴选、移译国外优秀教材上。经过几年的不懈努力，我们与Prentice Hall，Addison-Wesley，McGraw-Hill，Morgan Kaufmann等世界著名出版公司建立了良好的合作关系，从它们现有的数百种教材中甄选出Tanenbaum，Stroustrup，Kernighan，Jim Gray等大师名家的一批经典作品，以“计算机科学丛书”为总称出版，供读者学习、研究及庋藏。大理石纹理的封面，也正体现了这套丛书的品位和格调。

“计算机科学丛书”的出版工作得到了国内外学者的鼎力襄助，国内的专家不仅提供了中肯的选题指导，还不辞劳苦地担任了翻译和审校的工作；而原书的作者也相当关注其作品在中国的传播，有的还专诚为其书的中译本作序。迄今，“计算机科学丛书”已经出版了近百个品种，这些书籍在读者中树立了良好的口碑，并被许多高校采用为正式教材和参考书籍，为进一步推广与发展打下了坚实的基础。

随着学科建设的初步完善和教材改革的逐渐深化，教育界对国外计算机教材的需求和应用都步人一个新的阶段。为此，华章公司将加大引进教材的力度，在“华章教育”的总规划之下出版三个系列的计算机教材：针对本科生的核心课程，剔抉外版菁华而成“国外经典教材”系列；对影印版的教材，则单独开辟出“经典原版书库”；定位在高级教程和专业参考的“计算机科学丛书”还将保持原来的风格，继续出版新的品种。为了保证这三套丛书的权威性，同时也为了更好地为学校和老师们服务，华章公司聘请了中国科学院、北京大学、清华大学、国防科技大学、复旦大学、上海交通大学、南京大学、浙江大学、中国科技大学、哈尔滨工业大学、西安交通大学、中国人民大学、北京航空航天大学、北京邮电大学、中山大学、解放军理工大学、郑州大学、湖北工学院、中国国家信息安全测评认证中心等国内重点大学和科研机构在计算机的各个领域的著名学者组成“专家指导委员会”，为我们提供选题意见和出版监督。

“经典原版书库”是响应教育部提出的使用原版国外教材的号召，为国内高校的计算机教学度身订造的。在广泛地征求并听取丛书的“专家指导委员会”的意见后，我们最终选定了这30多种篇幅内容适度、讲解鞭辟入里的教材，其中的大部分已经被M.I.T.、Stanford、U.C. Berkley、C.M.U.等世界名牌大学采用。丛书不仅涵盖了程序设计、数据结构、操作系统、计算机体系结构、数据库、编译原理、软件工程、图形学、通信与网络、离散数学等国内大学计算机专业普遍开设的核心课程，而且各具特色——有的出自语言设计者之手、有的历三十年而不衰、有的已被全世界的几百所高校采用。在这些圆熟通博的名师大作的指引之下，读者必将在计算机科学的宫殿中由登堂而入室。

权威的作者、经典的教材、一流的译者、严格的审校、精细的编辑，这些因素使我们的图书有了质量的保证，但我们的目标是尽善尽美，而反馈的意见正是我们达到这一终极目标的重要帮助。教材的出版只是我们的后续服务的起点。华章公司欢迎老师和读者对我们的工作提出建议或给予指正，我们的联系方法如下：

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Preface

"Audience, level, and treatment — a description of such matters is what prefaces are supposed to be about."
— P. R. Halmos [173]

THIS BOOK IS BASED on a course of the same name that has been taught annually at Stanford University since 1970. About fifty students have taken it each year — juniors and seniors, but mostly graduate students — and alumni of these classes have begun to spawn similar courses elsewhere. Thus the time seems ripe to present the material to a wider audience (including sophomores).

It was a dark and stormy decade when Concrete Mathematics was born. Long-held values were constantly being questioned during those turbulent years; college campuses were hotbeds of controversy. The college curriculum itself was challenged, and mathematics did not escape scrutiny. John Hammersley had just written a thought-provoking article "On the enfeeblement of mathematical skills by 'Modern Mathematics' and by similar soft intellectual trash in schools and universities" [176]; other worried mathematicians [332] even asked, "Can mathematics be saved?" One of the present authors had embarked on a series of books called *The Art of Computer Programming*, and in writing the first volume he (DEK) had found that there were mathematical tools missing from his repertoire; the mathematics he needed for a thorough, well-grounded understanding of computer programs was quite different from what he'd learned as a mathematics major in college. So he introduced a new course, teaching what he wished somebody had taught him.

"People do acquire a little brief authority by equipping themselves with jargon: they can pontificate and air a superficial expertise. But what we should ask of educated mathematicians is not what they can speechify about, nor even what they know about the existing corpus of mathematical knowledge, but rather what can they now do with their learning and whether they can actually solve mathematical problems arising in practice. In short, we look for deeds not words."
— J. Hammersley [176]

The course title "Concrete Mathematics" was originally intended as an antidote to "Abstract Mathematics," since concrete classical results were rapidly being swept out of the modern mathematical curriculum by a new wave of abstract ideas popularly called the "New Math." Abstract mathematics is a wonderful subject, and there's nothing wrong with it: It's beautiful, general, and useful. But its adherents had become deluded that the rest of mathematics was inferior and no longer worthy of attention. The goal of generalization had become so fashionable that a generation of mathematicians had become unable to relish beauty in the particular, to enjoy the challenge of solving quantitative problems, or to appreciate the value of technique. Abstract mathematics was becoming inbred and losing touch with reality; mathematical education needed a concrete counterweight in order to restore a healthy balance.

When DEK taught Concrete Mathematics at Stanford for the first time, he explained the somewhat strange title by saying that it was his attempt to teach a math course that was hard instead of soft. He announced that, contrary to the expectations of some of his colleagues, he was *not* going to teach the Theory of Aggregates, nor Stone's Embedding Theorem, nor even the Stone–Čech compactification. (Several students from the civil engineering department got up and quietly left the room.)

"The heart of mathematics consists of concrete examples and concrete problems."
—P. R. Halmos [172]

Although Concrete Mathematics began as a reaction against other trends, the main reasons for its existence were positive instead of negative. And as the course continued its popular place in the curriculum, its subject matter "solidified" and proved to be valuable in a variety of new applications. Meanwhile, independent confirmation for the appropriateness of the name came from another direction, when Z. A. Melzak published two volumes entitled *Companion to Concrete Mathematics* [267].

"It is downright sinful to teach the abstract before the concrete."
—Z. A. Melzak [267]

The material of concrete mathematics may seem at first to be a disparate bag of tricks, but practice makes it into a disciplined set of tools. Indeed, the techniques have an underlying unity and a strong appeal for many people. When another one of the authors (RLG) first taught the course in 1979, the students had such fun that they decided to hold a class reunion a year later.

But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence relations, and to discover subtle patterns in data. You will be so fluent in algebraic techniques that you will often find it easier to obtain exact results than to settle for approximate answers that are valid only in a limiting sense.

Concrete Mathematics is a bridge to abstract mathematics.

The major topics treated in this book include sums, recurrences, elementary number theory, binomial coefficients, generating functions, discrete probability, and asymptotic methods. The emphasis is on manipulative technique rather than on existence theorems or combinatorial reasoning; the goal is for each reader to become as familiar with discrete operations (like the greatest-integer function and finite summation) as a student of calculus is familiar with continuous operations (like the absolute-value function and indefinite integration).

"The advanced reader who skips parts that appear too elementary may miss more than the less advanced reader who skips parts that appear too complex."
—G. Pólya [297]

Notice that this list of topics is quite different from what is usually taught nowadays in undergraduate courses entitled "Discrete Mathematics." Therefore the subject needs a distinctive name, and "Concrete Mathematics" has proved to be as suitable as any other.

(We're not bold enough to try Distinuous Mathematics.)

The original textbook for Stanford's course on concrete mathematics was the "Mathematical Preliminaries" section in *The Art of Computer Programming* [207]. But the presentation in those 110 pages is quite terse, so another author (OP) was inspired to draft a lengthy set of supplementary notes. The present book is an outgrowth of those notes; it is an expansion of, and a more leisurely introduction to, the material of Mathematical Preliminaries. Some of the more advanced parts have been omitted; on the other hand, several topics not found there have been included here so that the story will be complete.

"... a concrete life preserver thrown to students sinking in a sea of abstraction."
—W. Gottschalk

The authors have enjoyed putting this book together because the subject began to jell and to take on a life of its own before our eyes; this book almost seemed to write itself. Moreover, the somewhat unconventional approaches we have adopted in several places have seemed to fit together so well, after these years of experience, that we can't help feeling that this book is a kind of manifesto about our favorite way to do mathematics. So we think the book has turned out to be a tale of mathematical beauty and surprise, and we hope that our readers will share at least ϵ of the pleasure we had while writing it.

Since this book was born in a university setting, we have tried to capture the spirit of a contemporary classroom by adopting an informal style. Some people think that mathematics is a serious business that must always be cold and dry; but we think mathematics is fun, and we aren't ashamed to admit the fact. Why should a strict boundary line be drawn between work and play? Concrete mathematics is full of appealing patterns; the manipulations are not always easy, but the answers can be astonishingly attractive. The joys and sorrows of mathematical work are reflected explicitly in this book because they are part of our lives.

Math graffiti:
Kilroy wasn't Haar.
Free the group.
Nuke the kernel.
Power to the n.
$N=1 \Rightarrow P=NP$.

Students always know better than their teachers, so we have asked the first students of this material to contribute their frank opinions, as "graffiti" in the margins. Some of these marginal markings are merely corny, some are profound; some of them warn about ambiguities or obscurities, others are typical comments made by wise guys in the back row; some are positive, some are negative, some are zero. But they all are real indications of feelings that should make the text material easier to assimilate. (The inspiration for such marginal notes comes from a student handbook entitled *Approaching Stanford*, where the official university line is counterbalanced by the remarks of outgoing students. For example, Stanford says, "There are a few things you cannot miss in this amorphous shape which is Stanford"; the margin says, "Amorphous ... what the h*** does that mean? Typical of the pseudo-intellectualism around here." Stanford: "There is no end to the potential of a group of students living together." Graffito: "Stanford dorms are like zoos without a keeper.")

I have only a marginal interest in this subject.

This was the most enjoyable course I've ever had. But it might be nice to summarize the material as you go along.

The margins also include direct quotations from famous mathematicians of past generations, giving the actual words in which they announced some of their fundamental discoveries. Somehow it seems appropriate to mix the words of Leibniz, Euler, Gauss, and others with those of the people who will be continuing the work. Mathematics is an ongoing endeavor for people everywhere; many strands are being woven into one rich fabric.

This book contains more than 500 exercises, divided into six categories:

I see: Concrete mathematics means drilling.

- ***Warmups*** are exercises that EVERY READER should try to do when first reading the material.
- ***Basics*** are exercises to develop facts that are best learned by trying one's own derivation rather than by reading somebody else's.

The homework was tough but I learned a lot. It was worth every hour.

- ***Homework exercises*** are problems intended to deepen an understanding of material in the current chapter.
- ***Exam problems*** typically involve ideas from two or more chapters simultaneously; they are generally intended for use in take-home exams (not for in-class exams under time pressure).

Take-home exams are vital — keep them.

- ***Bonus problems*** go beyond what an average student of concrete mathematics is expected to handle while taking a course based on this book; they extend the text in interesting ways.

Exams were harder than the homework led me to expect.

- ***Research problems*** may or may not be humanly solvable, but the ones presented here seem to be worth a try (without time pressure).

Answers to all the exercises appear in Appendix A, often with additional information about related results. (Of course, the "answers" to research problems are incomplete; but even in these cases, partial results or hints are given that might prove to be helpful.) Readers are encouraged to look at the answers, especially the answers to the warmup problems, but only AFTER making a serious attempt to solve the problem without peeking.

Cheaters may pass this course by just copying the answers, but they're only cheating themselves.

We have tried in Appendix C to give proper credit to the sources of each exercise, since a great deal of creativity and/or luck often goes into the design of an instructive problem. Mathematicians have unfortunately developed a tradition of borrowing exercises without any acknowledgment; we believe that the opposite tradition, practiced for example by books and magazines about chess (where names, dates, and locations of original chess problems are routinely specified) is far superior. However, we have not been able to pin down the sources of many problems that have become part of the folklore. If any reader knows the origin of an exercise for which our citation is missing or inaccurate, we would be glad to learn the details so that we can correct the omission in subsequent editions of this book.

Difficult exams don't take into account students who have other classes to prepare for.

The typeface used for mathematics throughout this book is a new design by Hermann Zapf [227], commissioned by the American Mathematical Society and developed with the help of a committee that included B. Beeton, R. P. Boas, L. K. Durst, D. E. Knuth, P. Murdock, R. S. Palais, P. Renz, E. Swanson, S. B. Whidden, and W. B. Woolf. The underlying philosophy of Zapf's design is to capture the flavor of mathematics as it might be written by a mathematician with excellent handwriting. A handwritten rather than mechanical style is appropriate because people generally create mathematics with pen, pencil, or chalk. (For example, one of the trademarks of the new design is the symbol for zero, '0', which is slightly pointed at the top because a handwritten zero rarely closes together smoothly when the curve returns to its starting point.) The letters are upright, not italic, so that subscripts, superscripts, and accents are more easily fitted with ordinary symbols. This new type family has been named *AMS Euler*, after the great Swiss mathematician Leonhard Euler (1707–1783) who discovered so much of mathematics as we know it today. The alphabets include Euler Text (Aa Bb Cc through Xx Yy Zz), Euler Fraktur ($\mathfrak{A}\mathfrak{a}\,\mathfrak{B}\mathfrak{b}\,\mathfrak{C}\mathfrak{c}$ through $\mathfrak{X}\mathfrak{x}\,\mathfrak{Y}\mathfrak{y}\,\mathfrak{Z}\mathfrak{z}$), and Euler Script Capitals ($\mathcal{A}\,\mathcal{B}\,\mathcal{C}$ through $\mathcal{X}\,\mathcal{Y}\,\mathcal{Z}$), as well as Euler Greek ($\mathrm{A}\alpha\,\mathrm{B}\beta\,\Gamma\gamma$ through $\mathrm{X}\chi\,\Psi\psi\,\Omega\omega$) and special symbols such as $\wp$ and $\aleph$. We are especially pleased to be able to inaugurate the Euler family of typefaces in this book, because Leonhard Euler's spirit truly lives on every page: Concrete mathematics is Eulerian mathematics.

I'm unaccustomed to this face.

The authors are extremely grateful to Andrei Broder, Ernst Mayr, Andrew Yao, and Frances Yao, who contributed greatly to this book during the years that they taught Concrete Mathematics at Stanford. Furthermore we offer 1024 thanks to the teaching assistants who creatively transcribed what took place in class each year and who helped to design the examination questions; their names are listed in Appendix C. This book, which is essentially a compendium of sixteen years' worth of lecture notes, would have been impossible without their first-rate work.

Dear prof: Thanks for (1) the puns, (2) the subject matter.

Many other people have helped to make this book a reality. For example, we wish to commend the students at Brown, Columbia, CUNY, Princeton, Rice, and Stanford who contributed the choice graffiti and helped to debug our first drafts. Our contacts at Addison-Wesley were especially efficient and helpful; in particular, we wish to thank our publisher (Peter Gordon), production supervisor (Bette Aaronson), designer (Roy Brown), and copy editor (Lyn Dupré). The National Science Foundation and the Office of Naval Research have given invaluable support. Cheryl Graham was tremendously helpful as we prepared the index. And above all, we wish to thank our wives (Fan, Jill, and Amy) for their patience, support, encouragement, and ideas.

I don't see how what I've learned will ever help me.

This second edition features a new Section 5.8, which describes some important ideas that Doron Zeilberger discovered shortly after the first edition went to press. Additional improvements to the first printing can also be found on almost every page.

I had a lot of trouble in this class, but I know it sharpened my math skills and my thinking skills.

We have tried to produce a perfect book, but we are imperfect authors. Therefore we solicit help in correcting any mistakes that we've made. A reward of $2.56 will gratefully be paid to the first finder of any error, whether it is mathematical, historical, or typographical.

I would advise the casual student to stay away from this course.

Murray Hill, New Jersey —RLG
and Stanford, California DEK
May 1988 and October 1993 OP

A Note on Notation

SOME OF THE SYMBOLISM in this book has not (yet?) become standard. Here is a list of notations that might be unfamiliar to readers who have learned similar material from other books, together with the page numbers where these notations are explained. (See the general index, at the end of the book, for references to more standard notations.)

Notation	*Name*	*Page*
$\ln x$	natural logarithm: $\log_e x$	276
$\lg x$	binary logarithm: $\log_2 x$	70
$\log x$	common logarithm: $\log_{10} x$	449
$\lfloor x \rfloor$	floor: $\max\{n \mid n \leqslant x,\ \text{integer } n\}$	67
$\lceil x \rceil$	ceiling: $\min\{n \mid n \geqslant x,\ \text{integer } n\}$	67
$x \bmod y$	remainder: $x - y\lfloor x/y \rfloor$	82
$\{x\}$	fractional part: $x \bmod 1$	70
$\sum f(x)\,\delta x$	indefinite summation	48
$\sum_a^b f(x)\,\delta x$	definite summation	49
$x^{\underline{n}}$	falling factorial power: $x!/(x-n)!$	47, 211
$x^{\overline{n}}$	rising factorial power: $\Gamma(x+n)/\Gamma(x)$	48, 211
$n¡$	subfactorial: $n!/0! - n!/1! + \cdots + (-1)^n n!/n!$	194
$\Re z$	real part: x, if $z = x + iy$	64
$\Im z$	imaginary part: y, if $z = x + iy$	64
H_n	harmonic number: $1/1 + \cdots + 1/n$	29
$H_n^{(x)}$	generalized harmonic number: $1/1^x + \cdots + 1/n^x$	277

If you don't understand what the x denotes at the bottom of this page, try asking your Latin professor instead of your math professor.

$f^{(m)}(z)$	mth derivative of f at z	470
$\left[{n \atop m}\right]$	Stirling cycle number (the "first kind")	259
$\left\{{n \atop m}\right\}$	Stirling subset number (the "second kind")	258
$\left\langle{n \atop m}\right\rangle$	Eulerian number	267
$\left\langle\!\left\langle{n \atop m}\right\rangle\!\right\rangle$	Second-order Eulerian number	270
$(a_m \ldots a_0)_b$	radix notation for $\sum_{k=0}^{m} a_k b^k$	11
$K(a_1, \ldots, a_n)$	continuant polynomial	302
$F\left({a, b \atop c} \middle\vert z\right)$	hypergeometric function	205
$\#A$	cardinality: number of elements in the set A	39
$[z^n]\,f(z)$	coefficient of z^n in $f(z)$	197
$[\alpha \,..\, \beta]$	closed interval: the set $\{x \mid \alpha \leqslant x \leqslant \beta\}$	73
$[m = n]$	1 if $m = n$, otherwise 0*	24
$[m \backslash n]$	1 if m divides n, otherwise 0*	102
$[m \backslash\backslash n]$	1 if m exactly divides n, otherwise 0*	146
$[m \perp n]$	1 if m is relatively prime to n, otherwise 0*	115

Prestressed concrete mathematics is concrete mathematics that's preceded by a bewildering list of notations.

*In general, if S is any statement that can be true or false, the bracketed notation $[S]$ stands for 1 if S is true, 0 otherwise.

Throughout this text, we use single-quote marks ('...') to delimit text as it is *written*, double-quote marks ("...") for a phrase as it is *spoken*. Thus, the string of letters 'string' is sometimes called a "string."

Also 'nonstring' is a string.

An expression of the form 'a/bc' means the same as '$a/(bc)$'. Moreover, $\log x / \log y = (\log x)/(\log y)$ and $2n! = 2(n!)$.

Contents

1

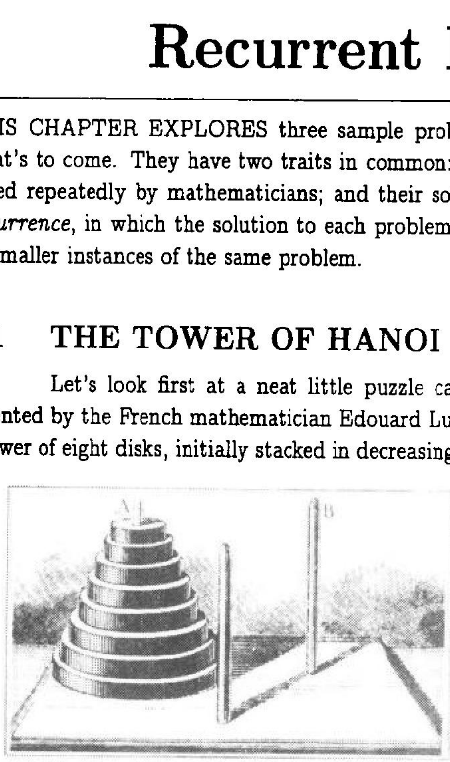
Recurrent Problems

THIS CHAPTER EXPLORES three sample problems that give a feel for what's to come. They have two traits in common: They've all been investigated repeatedly by mathematicians; and their solutions all use the idea of *recurrence*, in which the solution to each problem depends on the solutions to smaller instances of the same problem.

1.1 THE TOWER OF HANOI

Raise your hand if you've never seen this. OK, the rest of you can cut to equation (1.1).

Let's look first at a neat little puzzle called the Tower of Hanoi, invented by the French mathematician Edouard Lucas in 1883. We are given a tower of eight disks, initially stacked in decreasing size on one of three pegs:



The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger one onto a smaller.

Gold — wow. Are our disks made of concrete?

Lucas [260] furnished his toy with a romantic legend about a much larger Tower of Brahma, which supposedly has 64 disks of pure gold resting on three diamond needles. At the beginning of time, he said, God placed these golden disks on the first needle and ordained that a group of priests should transfer them to the third, according to the rules above. The priests reportedly work day and night at their task. When they finish, the Tower will crumble and the world will end.

It's not immediately obvious that the puzzle has a solution, but a little thought (or having seen the problem before) convinces us that it does. Now the question arises: What's the best we can do? That is, how many moves are necessary and sufficient to perform the task?

The best way to tackle a question like this is to generalize it a bit. The Tower of Brahma has 64 disks and the Tower of Hanoi has 8; let's consider what happens if there are n disks.

One advantage of this generalization is that we can scale the problem down even more. In fact, we'll see repeatedly in this book that it's advantageous to LOOK AT SMALL CASES first. It's easy to see how to transfer a tower that contains only one or two disks. And a small amount of experimentation shows how to transfer a tower of three.

The next step in solving the problem is to introduce appropriate notation: NAME AND CONQUER. Let's say that T_n is the minimum number of moves that will transfer n disks from one peg to another under Lucas's rules. Then T_1 is obviously 1, and $T_2 = 3$.

We can also get another piece of data for free, by considering the smallest case of all: Clearly $T_0 = 0$, because no moves at all are needed to transfer a tower of $n = 0$ disks! Smart mathematicians are not ashamed to think small, because general patterns are easier to perceive when the extreme cases are well understood (even when they are trivial).

But now let's change our perspective and try to think big; how can we transfer a large tower? Experiments with three disks show that the winning idea is to transfer the top two disks to the middle peg, then move the third, then bring the other two onto it. This gives us a clue for transferring n disks in general: We first transfer the $n-1$ smallest to a different peg (requiring T_{n-1} moves), then move the largest (requiring one move), and finally transfer the $n-1$ smallest back onto the largest (requiring another T_{n-1} moves). Thus we can transfer n disks (for $n > 0$) in at most $2T_{n-1}+1$ moves:

$$T_n \leqslant 2T_{n-1}+1\,, \qquad \text{for } n>0.$$

This formula uses '$\leqslant$' instead of '$=$' because our construction proves only that $2T_{n-1}+1$ moves suffice; we haven't shown that $2T_{n-1}+1$ moves are necessary. A clever person might be able to think of a shortcut.

But is there a better way? Actually no. At some point we must move the largest disk. When we do, the $n-1$ smallest must be on a single peg, and it has taken at least T_{n-1} moves to put them there. We might move the largest disk more than once, if we're not too alert. But after moving the largest disk for the last time, we must transfer the $n-1$ smallest disks (which must again be on a single peg) back onto the largest; this too requires T_{n-1} moves. Hence

Most of the published "solutions" to Lucas's problem, like the early one of Allardice and Fraser [7], fail to explain why T_n must be $\geqslant 2T_{n-1}+1$.

$$T_n \geqslant 2T_{n-1}+1\,, \qquad \text{for } n>0.$$