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Proceedings of the First Los Alamos Conference
on Nonlinear Problems,
Los Alamos, NM, U.S.A., March 2-6 1981

Edited by

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PREFACE

Recognizing the growing awareness of common problems and answers in the nonlinear sciences, the Los Alamos Center for Nonlinear Studies (CNLS) was established by the Laboratory Director in October 1980, to coordinate interdisciplinary studies, to strengthen ties among Laboratory researchers and with the academic community, and to interface between basic and applied areas. The breadth of research problems at Los Alamos and the variety of technical expertise reflected in its staff offers a fertile environment for the CNLS. But from its inception, the Center was intended not for Los Alamos alone, but also as an international resource for the entire nonlinear community.

To realize this intention, the CNLS, through its Chairman, Alwyn C. Scott, has developed several continuing modes of operation:

- identifying broad themes for intensive interdisciplinary research (currently, these themes are nonlinear phenomena in reactive flows and chaos and coherence in physical systems);
- coordinating an active visitor program, including both guest lecture series and longterm collaborative visits;
- sponsoring technical workshops aimed at bringing experts together to expound recent results and delineate future directions in specific nonlinear problems (recent workshops have included "Adaptive Grid Methods," "Coupled Nonlinear Oscillators," and "Solitons and the Bethe Ansatz"); and
- hosting an annual international conference on some topical area of nonlinear science.

In the last category, the major event for the CNLS in its first year was an international conference on 'Nonlinear Problems: Present and Future' held at the Los Alamos National Security and Resources Study Center, Los Alamos, March 2-6, 1981, chaired by Mark Kac and Stanislav Ulam. This volume contains the edited proceedings of that conference. We are honored to dedicate the proceedings to Fermi, Pasta, and Ulam, distinguished Los Alamos alumni who have set a tradition of excellence to which the CNLS must aspire. [We have all been saddened by the death of John Pasta since the Conference took place and hope that this volume will be accepted as our small tribute to his imaginative career.]

As befits an inaugral conference, a very wide spectrum of topics were represented ranging from pure mathematics, through numerical methods, to sophisticated experiments on fluids and solids. More specialized meetings are anticipated in future years including some of the many topics that it was impossible to cover in 1981. However, the deliberately interdisciplinary atmosphere of the inaugral meeting truly reflected an exciting stage of development of nonlinear science as a unified subject. The provocative title attracted well over two hundred participants and a distinguished list of speakers, many of whom succeeded admirably in overcoming scientific language barriers and generating a broad interest in their fundamental problems. Four major topics were reperesented through survey lectures and workshop activities: turbulence in plasmas and fluids (both the onset and fully-developed turbulence); nonlinearity in field theory and in low-dimensional solids; reaction- diffusion processes; and new methods in nonlinear mathematics. As described in detail in the Contents, we have preserved these divisions in the Proceedings, supplementing invited papers with a small number of relevant contributed ones. It is our firm impression that these articles, through survey and original work, represent the cutting edges of several important areas of nonlinearity. We hope they will be valuable reading for novices and experts alike.

PREFACE

We think that all those who attended the Conference will remember it for its stimulation and unobtrusive organization. This crucial combination could not have been achieved without the advice and support of our colleagues in the CNLS. The Director and his efficient staff provided every conceivable help in coordinating the splendid Laboratory facilities. Mark Kac and Stanislav Ulam were supportive conference chairman and Stanislav Ulam graciously agreed to give a nostalgic after-dinner speech on the FPU problem which was admired by all. No conference is better than its secretary. In Janet Gerwin we had a secretary whose competence was apparent at every stage before, during, and after the conference. All three organizers are immeasurably indebted to her for her skill in the face of the continual crises! We are also happy to thank Janet, Chris Davis, Frankie Gomez, Mary Plehn, and Kate Procknow for their skillful assistance in preparing these proceedings. Last but not least we must thank our publishers for their excellent cooperation and every conference attendee for joining us in this celebration of nonlinear physics.

Los Alamos

A. R. Bishop
D. K. Campbell
B. Nicolaenko

DEDICATION

It started with an experiment--the new kind in which the same instrument, the new computer, both creates and probes an idealization of the real world.

The purpose, quite modest, was to test what seemed beyond doubt, namely, that in a nonlinear discretized string the energy initially concentrated in one vibrational mode ultimately distributes itself among all modes. The result, first announced in a Los Alamos report, was however, startlingly different: after an initial tendency toward equipartition, the energy flowed back to the initial mode.

Thus a new chapter of nonlinear science began and much of its spectacular growth in the past quarter of a century is directly or indirectly traceable to the pioneering experiment with the nonlinear string.

It is therefore fitting that this Conference marking the creation of the Los Alamos Center for Nonlinear Studies be dedicated to the authors of that historic 1955 report: Fermi, Pasta, and Ulam.

Los Alamos

M. Kac
N. Metropolis

TABLE OF CONTENTS

TABLE OF CONTENTS

PART I
New Methods and Results in Nonlinear Analysis

Nonlinear Problems: Present and Future
A.R. Bishop, D.K. Campbell, B. Nicolaenko (eds.)

OPTIMAL CONTROL OF NON WELL POSED DISTRIBUTED SYSTEMS AND RELATED NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Jacques-Louis LIONS

Collège de France and INRIA

Various practical problems lead to the question of optimal control of non well posed systems - such as non linear unstable systems. We begin with simple linear - quadratic examples of non well posed evolution systems ; the optimality system leads to new non linear Partial Differential Equations of Ricatti's type. We study then non linear parabolic unstable systems, with a distributed or a boundary control, with or without constraints. The optimality system is given.

INTRODUCTION

1. Let $\mathcal{A}$ be a partial differential operator, linear or not, of evolution or of stationary type. In the usual theory of optimal control of distributed systems, the state equation is given, in a formal manner, by

$$\mathcal{A} y = \mathcal{B} v \tag{1}$$

where v denotes the control variable (or function) and $\mathcal{B}$ is an operator which can be thought of as giving boundary conditions ; in (1) one has to add initial conditions if $\mathcal{A}$ is of evolution type.

In the usual theory one assumes that, given v in a suitable Banach space U , equation (1) subject to appropriate boundary and initial conditions, admits a unique solution denoted by y(v) ; y(v) is the state of the system ; y(v) belongs to a space Y when v spans U.

Then the cost function is given by

$$J(v) = \phi(y(v)) + \psi(\|v\|_U) \tag{2}$$

where ϕ is a continuous functional from $Y \to \mathbb{R}$, where $\|v\|_U$ denotes the norm of v in U and where $\lambda \to \psi(\lambda)$ is continuous for $\lambda \geq 0$, $\psi(0) = 0$, $\psi(\lambda) \to +\infty$ as $\lambda \to +\infty$.

If U_{ad} denotes a (suitable) subset of U, then the problem is to find

$$\inf J(v), \ v \in U_{ad}. \tag{3}$$

If (3) admits a solution u, one of the main questions is then to find a set of necessary (or necessary and sufficient) conditions for characterizing u, i.e. to find the optimality system. For these questions we refer to J.L. LIONS [1][2][3] and to the Bibliography therein.

2. A slightly different situation can occur if the functional ϕ in (2) is not defined on the whole space Y. Then one has to introduce new functional spaces.

et us give an example. Let Ω be a bounded open set of $\mathbb{R}^3$ with boundary Γ ; we onsider the state equation

$$\frac{\partial y}{\partial t} - \Delta y = v(t)\delta(x-b) \text{ in } Q = \Omega \times]0,T[$$

$$y = 0 \text{ on } \Sigma = \Gamma \times]0,T[, \qquad (4)$$

$$y(x,0) = 0$$

here $\delta(x-b)$ denotes the Dirac measure at point $b \in \Omega$.

iven $v \in L^2(0,T)$, problem (4) admits a unique weak solution (cf. J.L. LIONS and MAGENES [1]) $y(v) \in L^2(Q)$.

et us consider now the cost function

$$J(v) = \int_\Omega [y(x,T;v)-z_d]^2 \, dx + N \int_0^T v^2 \, dt, \qquad (5)$$

here z_d is given in $L^2(\Omega)$ and $N > 0$. In general, for $v \in L^2(0,T)$, $y(\cdot,T;v)$ is defied but as an element of $H^{-1}(\Omega)$ (Sobolev space of order -1) and not an element of (Ω). Therefore (5) does not make sense for $v \in L^2(0,T)$. One has then to restrict to those v's such that

$$v \in L^2(0,T) \text{ and } y(\cdot,T;v) \in L^2(\Omega). \qquad (6)$$

his defines a Hilbert space (when provided with the norm $(\int_0^T v^2 dt + \int_\Omega y(x,T;v)^2 dx^{1/2}$ y U, and if

$$U_{ad} \subset U \, , \qquad (7)$$

consider again problem (3). In order to proceed it is necessary to study U, not ly to make things more precise but also because the dual U' of U is needed for iting the optimality system. One verifies that U coincides with the set of those s in $L^2(0,T)$ such that

$$\int_0^T \int_0^T (2T-(t+s))^{-3/2} \, v(t)v(x) dt \, ds < \infty. \qquad (8)$$

or questions of this type we refer to J.L. LIONS [4][5], J. SIMON [1].

A third situation can occur when (1) is not a well posed problem. Equations (1) which have a physical interest and which lead to non well set problems arise n unstable phenomena, in situations where we have bifurcations - cf. J.P. KERNEVEZ L. LIONS and D. THOMAS [1]. One has then to change significantly the point of ew . One considers the set of v and z such that

$$v \in U, \; z \in Y \qquad (9)$$

$$\mathcal{A} z = \mathcal{B} v \, . \qquad (10)$$

hen one considers the cost function

$$J(v,z) = \phi(z) + \psi(\|v\|_U) \qquad (11)$$

nd one looks for

$$\inf J(v,z), \quad v,z \text{ subject to (9) (10)} \qquad (12)$$

ith the possible added constraint

$v \in U_{ad}$. ∎ (13)

As an example (without physical interest) we consider

$$\begin{cases} \frac{\partial z}{\partial t} + \Delta z = v \text{ in } Q \\ v, z \in L^2(Q) \end{cases} \tag{14}$$

with the conditions

$$z(x,0) = 0 \ , \ z = 0 \text{ on } \Sigma \tag{15}$$

(one can prove that conditions (15) do make sense ; cf. Section 1 below). Let the cost function be given by

$$J(v,z) = \|z-z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)} \tag{16}$$

and let U_{ad} be a closed convex subset of $L^2(Q)$ such that the set of those v,z's such that $v \in U_{ad}$ and (14) (15) hold true is not empty. Then

$$\inf J(v,z) \ , \quad v \in U_{ad} \ , \ v,z \text{ satisfy (14) (15)} \tag{17}$$

admits a unique solution $\{u,y\}$. ∎

Returning to the general case, we want to find an optimality system for these problems of optimal control.

4. We consider in this paper three (of the many) situations of such problems.
In Section 1 we consider a system of type (14) but which is also non well posed for $t < T$, namely

$$\frac{\partial z}{\partial t} + m(t)\,\Delta z = v, \tag{18}$$

with $m > 0$ (resp. < 0) near 0 (resp. near T).

Decoupling the optimality system leads to apparently new nonlinear Partial Differential equations.

In Section 2 we consider unstable systems governed by

$$\frac{\partial z}{\partial t} - \Delta z - z^3 = v \tag{19}$$

(or with boundary control).

Other situations are indicated in J.L. LIONS [5], such as the case of elliptic systems which can be controlled by Cauchy data on part of the boundary.

Problems of optimum design where again the state equation is not well set will be studied elsewhere.

5. Existence problems for not necessarily well set problems (such as Navier-Stokes equations in space dimension equal 3) have been studied by A.V. FOURSIKOV [1] ; this author does not consider the optimality system.

The optimality system for problem (17) involves new functional spaces (of distributions of infinite order) ; we refer to P. RIVERA [1].

1. New non linear Partial Differential equation of Riccati's type.

1.1. Setting of the problem.

We consider couples $\{v,z\}$ such that

$$v,z \in L^2(Q) \times L^2(Q) \quad , \; Q = \Omega \times]0,T[, \tag{1.1}$$

and

$$\frac{\partial z}{\partial t} + m(t)\, z = v \text{ in } Q, \tag{1.2}$$

$$z(x,0) = 0 \text{ in } \Omega, \tag{1.3}$$

$$z = 0 \text{ on } \Sigma = \Gamma \times]0,T[. \tag{1.4}$$

In (1.2) m denotes a continuous fonction with a graph as represented on Fig. 1.

Conditions (1.3) (1.4) make sense.
Let us check it for (1.3) ; it follows from (1.1) (1.2) that

$$\begin{cases} z \in L^2(0,T;L^2(\Omega)), \\ \dfrac{\partial z}{\partial t} \in L^2(0,T;H^{-2}(\Omega)) \end{cases} \tag{1.5}$$

(where $H^{-2}(\Omega)$ = dual of $H_0^2(\Omega)$, $H_0^2(\Omega) = \{\phi \mid \phi, \frac{\partial \phi}{\partial x_i}, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \in L^2(\Omega), \phi = 0, \frac{\partial \phi}{\partial x_i} = 0 \text{ on } \Gamma\}$.) ; it follows from (1.5) and from standard results that z is continuous form $[0,T] \to H^{-1}(\Omega)$ so that (1.3) makes sense. One checks by similar techniques that (1.4) makes sense.

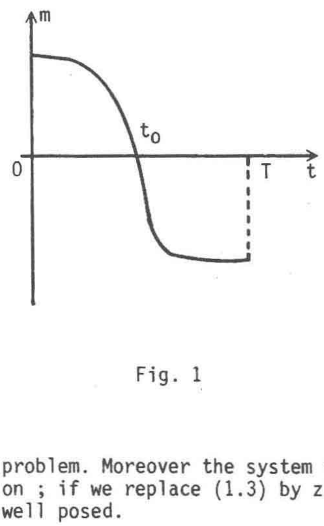

Fig. 1

Of course (1.2) (1.3) (1.4) is a non well posed problem. Moreover the system is also non well posed in the backward time direction ; if we replace (1.3) by $z(x,T) = 0$ in Ω, the corresponding problem is also non well posed.

The cost function that we consider is given by

$$J(v,z) = \|z-z_d\|^2_{L^2(Q)} + N\,\|v\|^2_{L^2(Q)} \quad , \tag{1.6}$$

z_d given in $L^2(Q)$, $N > 0$.

We consider the "no-constraints" problem, namely

$$\inf J(v,z) \; , \; v,z \text{ subject to } (1.1)\ldots(1.4). \tag{1.7}$$

1.2. Optimality system.

It is a simple matter to check that problem (1.7) admits a unique solution u,y. We want to characterize u,y ; this characterization is given by the following result : there exists a unique set $\{u,y,p\}$ of functions such that

$$\frac{\partial y}{\partial t} + m(t)\Delta y = u \ , \ -\frac{\partial p}{\partial t} + m(t)\Delta p = y - z_d \ , \tag{1.8}$$

$$y = p = 0 \text{ on } \Sigma, \tag{1.9}$$

$$y(x,0) = 0, \ p(x,T) = 0 \tag{1.10}$$

$$y,p \in L^2(Q), \tag{1.11}$$

$$p + Nu = 0 \tag{1.12}$$

One can of course eliminate u by (1.8) (1.12). The system in u,y,p is called the optimality system ; p is a Lagrange multiplier.

The proof of this result can be obtained as follows ; one considers the penalized problem

$$J_\varepsilon(v,z) = \|z - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)} + \frac{1}{\varepsilon}\left\|\frac{\partial z}{\partial t} + m(t)\Delta z - v\right\|^2_{L^2(Q)} \tag{1.13}$$

where $v \in L^2(Q)$, $z \in L^2(Q)$, $\frac{\partial z}{\partial t} + m(t)\Delta z \in L^2(Q)$ and $z(x,0) = 0$; $z = 0$ in Σ, and where $\varepsilon > 0$ is "small".

Then

$$\inf J_\varepsilon(v,z) = J_\varepsilon(u_\varepsilon, y_\varepsilon) . \tag{1.14}$$

One defines p_ε by

$$p_\varepsilon = -\frac{1}{\varepsilon}\left(\frac{\partial y_\varepsilon}{\partial t} + m\Delta y_\varepsilon - u_\varepsilon\right) . \tag{1.15}$$

One verifies that

$$\begin{cases} -\dfrac{\partial p_\varepsilon}{\partial t} + m(t)\Delta p_\varepsilon = y_\varepsilon - z_d \text{ in } Q, \\ p_\varepsilon(x,T) = 0 \ , \ p_\varepsilon = 0 \text{ on } \Sigma \end{cases} \tag{1.16}$$

and that

$$p_\varepsilon + Nu_\varepsilon = 0 \text{ in } Q. \tag{1.17}$$

It follows from (1.13) and (1.17) that

$$u_\varepsilon, y_\varepsilon, p_\varepsilon \text{ remain , as } \varepsilon \to 0, \text{ in a bounded set of } (L^2(Q))^3, \tag{1.18}$$

and that

$$\frac{\partial y_\varepsilon}{\partial t} + m(t)\Delta y_\varepsilon - u_\varepsilon = \sqrt{\varepsilon}\, g_\varepsilon \ , \ \|g_\varepsilon\|_{L^2(Q)} \leq C \tag{1.19}$$

Therefore one can pass to the limit in $\varepsilon \to 0$. One verifies that $u_\varepsilon, y_\varepsilon \to u,y$ in $L^2(Q) \times L^2(Q)$ and that u,y,p satisfy the optimality system.

One can write a variational formulation for the problem, as follows : p is the solution of

$$\left(-\frac{\partial p}{\partial t} + m(t)\Delta p, -\frac{\partial q}{\partial t} + m(t)\Delta q\right) + \frac{1}{N}(p,q) = -\left(z_d, -\frac{\partial q}{\partial t} + m\Delta q\right) \quad \forall q \tag{1.20}$$

(where $(p,q) = \int_Q p\, q\, dx\, dt$) where $q \in L^2(Q)$, $-\frac{\partial q}{\partial t} + m(t)\Delta q \in L^2(Q)$, $q(x,T) = 0$,

$q = 0$ on Σ. ∎

1.3. We are now going to show how to "uncouple" the optimality system.

The technique is similar to the one in J.L. LIONS [1] but one deals now with weak solutions, due to the fact that the "state equation" is not well posed.

One considers a problem analogous to (1.8)...(1.12) but in the interval $]s,T[$, $0 < s < T$:

$$\begin{cases} \frac{\partial\phi}{\partial t} + m\Delta\phi + \frac{1}{N}\psi = 0 \ , \quad -\frac{\partial\psi}{\partial t} + m\Delta\psi = \phi - z_d \ , \text{ in } \Omega\times]s,T[\ , \\ \phi(x,s) = h(x) \text{ in } \Omega, \ \psi(x,T) = 0 \text{ in } \Omega, \\ \phi = \psi = 0 \text{ on } \Gamma\times]s,T[. \end{cases} \tag{1.21}$$

In (1.21) h is given in, say, the space of C^∞ smooth functions with compact support. Problem (1.21) is the optimality system for a problem entirely analogous to (1.6) (1.7) but with $\Omega\times]0,T[$ replaced by $\Omega\times]s,T[$ and with $z(x,0) = 0$ replaced by $\phi(x,s) = h(x)$. Therefore (1.21) admits a unique solution and

$$\psi(x,s) \text{ is uniquely defined.} \tag{1.22}$$

The space where $\psi(\cdot,s)$ belongs depends whether $s < t_0$ or $s > t_0$. If $s < t_0$, one deals with a well set system, and $\psi(\cdot,s) \in H_0^1(\Omega)$; if $s < t_0$, one has to work with weak solutions and $\psi(\cdot,s) \in H^{-1}(\Omega)$.

We have :

$$\psi(\cdot,s) = P(s)h + r(s) \tag{1.23}$$

where $P(s)$ is a linear operator ; one has

$$\begin{cases} P(s) \in \mathcal{L}(H^{-1}(\Omega) \ ; \ H^{-1}(\Omega)) \text{ if } s \leq t_0 \ , \\ P(s) \in \mathcal{L}(H^{-1}(\Omega) \ ; \ H_0^1(\Omega)) \quad \text{if } s > t_0 . \end{cases} \tag{1.24}$$

If we take in (1.21) $h = y(\cdot,s) = y(s)$ then ϕ,ψ = restriction of y,p to $\Omega\times]s,T[$, so that (1.23) becomes (changing s into t) :

$$p(t) = P(t)y(t) + r(t) . \tag{1.25}$$

Using the L. Schwartz kernel theorem [1], one sees that

$$P(t)h = \int_\Omega P(x,\xi,t) \ h(\xi) \ d\xi \tag{1.26}$$

where the kernel $P(x,\xi,t)$ is a distribution on $\Omega_x \times \Omega_\xi$.
Using (1.25) into (1.8)...(1.12) one obtains finally that $P(x,\xi,t)$ satisfies

$$-\frac{\partial P}{\partial t} + m(t)(\Delta_x + \Delta_\xi)P + \frac{1}{N}\int_\Omega P(x,\zeta,t)P(\zeta,\xi,t)d\zeta = \delta(x-\xi), \tag{1.27}$$

$$P(x,\xi,T) = 0 \tag{1.28}$$

and

$$P(x,\xi,t) = P(\xi,x,t) \quad \forall x,\xi \in \Omega\times\Omega . \tag{1.29}$$

The Boundary conditions are of Dirichlet type, in the usual sense for $t_0 < t < T$