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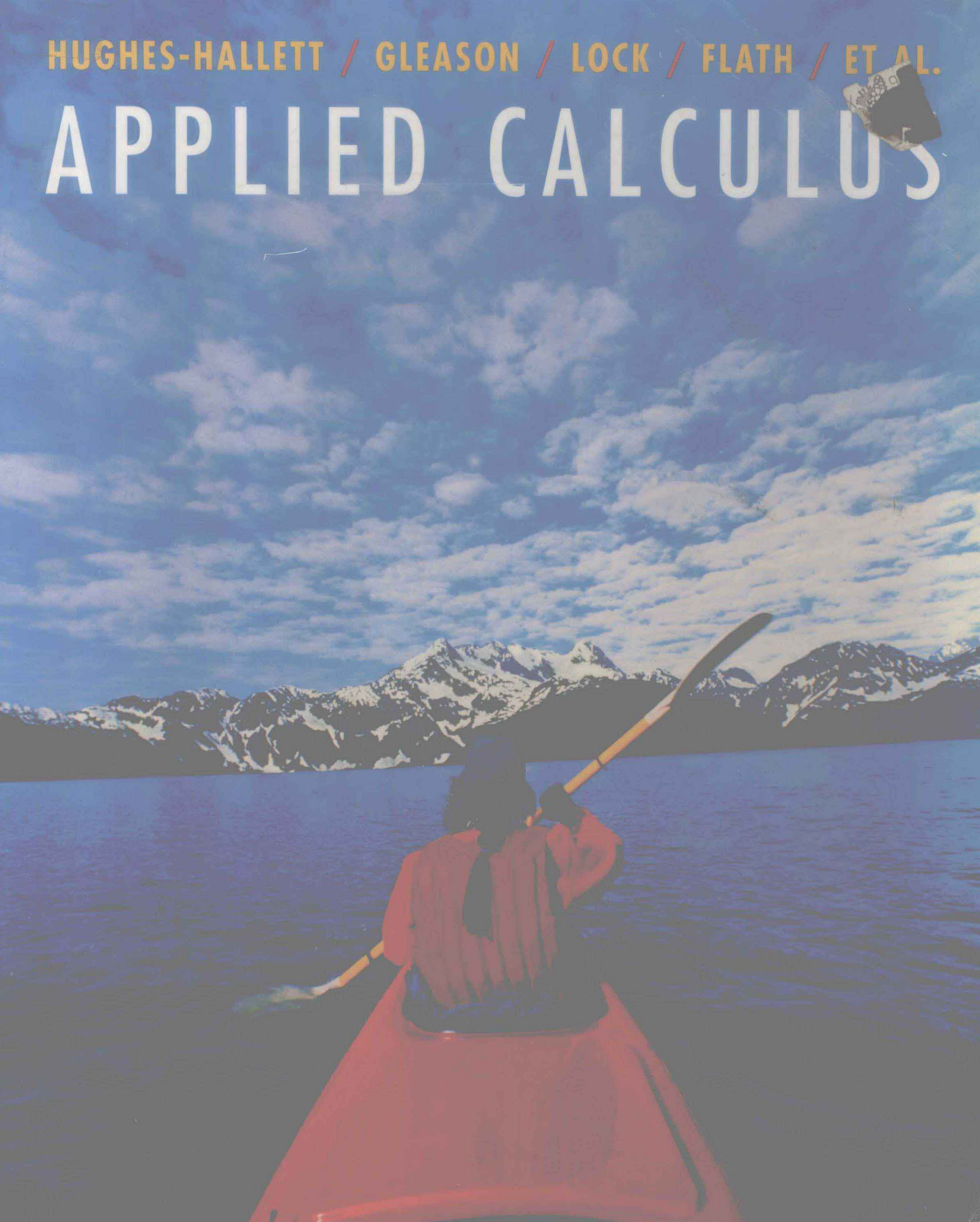
APPLIED CALCULUS



APPLIED CALCULUS

Produced by the Consortium based at Harvard which was formed under a National Science Foundation Grant.

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Dedicated to Robin, Kari, Eric, and Dennis,
and
Laura, Hannah, Ben, Natty, Isaac, and Matthias.

PREFACE

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but the rules and procedures—thereby losing sight of both the mathematics and of its practical value. This edition of *Applied Calculus* continues our effort to refocus the teaching of calculus on concepts as well as procedures.

A Focused Vision: Conceptual Understanding

Our goal is to provide students with a clear understanding of the ideas of calculus as a solid foundation for subsequent courses. We began work on this book by talking to faculty in business, economics, biology, and a wide range of other fields, as well as to many mathematicians who teach applied calculus. As a result of these discussions we included some new topics and omitted some traditional topics whose inclusion we could not justify. In the process, we also changed the focus of certain topics.

The First Edition: Expanded Choices

The first edition has the same vision as the preliminary edition and provides instructors with additional choices through the *Focus On* sections. Instructors can select a focus for their course which reflects their interests and the needs of their students. In particular:

- All *Focus On* sections are optional.
- Chapter 3 (the definite integral) may be covered immediately before Chapter 6 (using the integral).
- For an earlier start on calculus, Chapter 2 and Chapter 3 may be covered after Section 1.6.
- In Chapters 5 and 6, instructors may select the sections relevant to their students.
- Chapters 7 (multivariable calculus) and 8 (differential equations) may be covered in either order.

Because different users often choose very different topics to cover in a one-semester applied calculus course, we have designed this book for either a one-semester course (with a lot of flexibility in choosing topics) or a two-semester course. Sample syllabi outlining various options are provided in the Instructor's Manual.

Guiding Principles: Varied Problems and the Rule of Four

Since students usually learn most when they are active, we feel that the exercises in a text are of central importance. In addition, we have found that multiple representations encourage students to reflect on the meaning of the material. Consequently, we have been guided by the following principles.

- Our problems are varied. Some are very straightforward and some are challenging. Most require students to understand the concepts and cannot be done by following a template in the text.
- The Rule of Four: Where appropriate, topics should be presented geometrically, numerically, analytically, and verbally.

What Student Background is Expected?

This book is intended for students in business, the social sciences, and the life sciences. We have found the material to be thought-provoking for well-prepared students while still accessible to students with weak algebra backgrounds. Providing numerical and graphical approaches as well as the algebraic gives students several ways of mastering the material. This approach encourages students to persist, thereby lowering failure rates.

Technology

We take advantage of computers and graphing calculators to help students learn to think mathematically. For example, using a graphing calculator to zoom in on functions is an excellent way of seeing local linearity. The ability to use technology effectively as a tool is important. Students are expected to use their own judgement to determine where technology is useful.

However, the book does not require any specific software or technology. Instructors have used the materials with graphing calculators, graphing software, and computer algebra systems. Any technology with the ability to graph functions and perform numerical integration will suffice.

Content

This content represents our vision of how applied calculus can be taught. It is flexible enough to accommodate individual course needs and requirements. Topics can easily be added or deleted, or the order changed.

Chapter 1: Functions and Change

Chapter 1 introduces the concept of a function and the idea of change, including the distinction between total change and rate of change. All elementary functions are introduced here. Although the functions are probably familiar, the graphical, numerical, verbal, and modeling approach to them is likely to be new. We introduce exponential functions early, since they are fundamental to the understanding of real-world processes.

Focus on Modeling: The first section introduces the student to fitting formulas to data and the second section provides further discussion of compound interest and the definition of the number e.

Chapter 2: Rate of Change: The Derivative

Chapter 2 presents the key concept of the derivative according to the Rule of Four. The purpose of this chapter is to give the student a practical understanding of the meaning of the derivative and its interpretation as an instantaneous rate of change. After finishing this chapter, a student will be able to find derivatives numerically (by taking arbitrarily fine difference quotients), visualize derivatives graphically as the slope of the graph, and interpret the meaning of first and second derivatives in various applications. The student will also understand the concept of marginality and recognize the derivative as a function in its own right.

Focus on Theory: This section discusses limits and continuity and presents the symbolic definition of the derivative.

Chapter 3: Accumulated Change: The Definite Integral

Chapter 3 presents the key concept of the definite integral, along the same lines as Chapter 2. Chapter 3 can be delayed until after Chapter 5 without difficulty.

The purpose of this chapter is to give the student a practical understanding of the definite integral as a limit of Riemann sums, and to bring out the connection between the derivative and the definite integral in the Fundamental Theorem of Calculus. We use the same method as in Chapter 2, introducing the fundamental concept in depth without going into technique. The student will finish the chapter with a good grasp of the definite integral as a limit of Riemann sums, with the ability to approximate a definite integral numerically, and with an understanding of how to interpret the definite integral in various contexts.

Focus on Theory: This section presents the Second Fundamental Theorem of Calculus and the properties of the definite integral.

Chapter 4: Short-Cuts to Differentiation

The derivatives of all the functions in Chapter 1 are introduced, as well as the rules for differentiating products, quotients, and composite functions.

Focus on Theory: This section uses the definition of the derivative to obtain all the differentiation rules.

Focus on Practice: This section provides a collection of differentiation problems for skill building.

Chapter 5: Using the Derivative

The aim of this chapter is to enable the student to use the derivative in solving problems, including optimization and graphing. It is not necessary to cover all the sections.

Chapter 6: Using the Integral

This chapter presents applications of the definite integral. It is not meant to be comprehensive and it is not necessary to cover all the sections. This chapter includes antiderivatives and a discussion of probability.

Focus on Practice: This section provides a collection of integration problems for skill building.

Chapter 7: Functions of Several Variables

Chapter 7 introduces functions of two variables from several points of view, using contour diagrams, formulas, and tables. It gives students the skills to read contour diagrams and think graphically, to read tables and think numerically, and to apply these skills, along with their algebraic skills, to modeling. The idea of the partial derivative is introduced from graphical, numerical, and analytical viewpoints. Partial derivatives are then applied to optimization problems, ending with a discussion of Lagrange multipliers.

Focus on Theory: This section uses optimization to derive the formula for the regression line.

Chapter 8: Differential Equations

This chapter introduces differential equations. The emphasis is on qualitative solutions, modeling, and interpretation. This chapter includes applications of systems of differential equations to population models, the spread of disease, and predator-prey interactions.

Changes from the Preliminary Editions

The preliminary editions were used by a large and diverse group of schools in semester and quarter systems, in large lectures and small classes, in computer labs, small groups, and traditional settings, and with a number of different technologies. In preparing this edition, we solicited comments from a large number of mathematicians who had used the text. We continued to discuss with our colleagues in client disciplines the mathematical needs of their students. We were offered many valuable suggestions, which we have tried to incorporate, while maintaining our original commitment to a focused treatment of a limited number of topics.

General Changes

- More applications to the life sciences, business, economics, and the social sciences have been included throughout.
- There are more easy and medium level problems in each section, and drill problems in the *Focus on Practice* sections at the end of Chapter 4 (differentiation) and Chapter 6 (integration).
- There are short answers in the back of the book to odd-numbered problems where appropriate.
- Each chapter concludes with a review of the main points.
- Two projects are included at the end of each chapter; a selection of projects which involve the use of spreadsheets is in an appendix.

Chapter-by-Chapter Description of Changes from Applied Calculus Preliminary Edition

- Chapter 1. A new section is included at the beginning of the chapter introducing change and rate of change, and the theme of change runs throughout the chapter. The section on power functions comes earlier and has been shortened to give it greater emphasis on applications and proportionality. The material on compound interest and fitting formulas to data has been moved to the Focus on Modeling section at the end of the chapter.
- Chapter 2. The material on average rate of change has moved to Chapter 1. The symbolic definition of the derivative has been moved to the Focus on Theory section at the end of the chapter. The material on average cost has been moved to Chapter 5.

- Chapter 3. The concept of accumulated change is developed from a variety of applications rather than focusing exclusively on distance traveled. The section on average value has been moved to Chapter 6. The section on the Fundamental Theorem of Calculus has been rewritten and a Focus on Theory section has been added which introduces the Second Fundamental Theorem of Calculus and the properties of the definite integral.
- Chapter 4. The derivations of the formulas have been moved to the Focus on Theory section at the end of the chapter. The material on the derivatives of periodic functions has been moved to the last section.
- Chapter 5. The section on families of curves has been replaced with sections on the surge function and the logistic function. The material on optimization has been expanded, and the introduction to average cost has been moved to this chapter.
- Chapter 6. The section on average value has been moved to this chapter. Material has been added on finding and using antiderivatives and on improper integrals. A section on relative growth rates has been added and the material on probability and distributions has been simplified.
- Chapter 7. This section combines Chapter 8 and Chapter 9 from Applied Calculus Preliminary Edition. The material has been streamlined and simplified. The sections on surfaces have been deleted and the material on contour diagrams has been expanded.
- Chapter 8. This comes from Chapter 7 of Applied Calculus Preliminary Edition. The emphasis on applications and modeling has been expanded and sections on systems of differential equations have been added.

Chapter-by-Chapter Changes from Brief Calculus Preliminary Edition

- Chapter 1. The first four sections of Chapter 4 in Brief Calculus Preliminary Edition have been moved to the end of Chapter 1. The section on power functions comes earlier and has been shortened to give it greater emphasis on applications and proportionality. Material on composition of functions has been added. The material on fitting formulas to data has been moved to the Focus on Modeling section at the end of the chapter.
- Chapter 2. The symbolic definition of the derivative and additional material on limits are included in the Focus on Theory section at the end of the chapter.
- Chapter 3. The section on average value has been moved to Chapter 6. The section on the Fundamental Theorem of Calculus has been rewritten and a Focus on Theory section has been added which introduces the Second Fundamental Theorem of Calculus and the properties of the definite integral.
- Chapter 4. This chapter now covers short-cuts to differentiation. The section on antiderivatives has been moved to Chapter 6. The section on deriving the formulas has been moved to the Focus on Theory section at the end of the chapter.
- Chapter 5. This chapter now covers applications of the derivative. It includes the material on the surge function and the logistic function that used to be in Chapter 4. New sections on average cost and elasticity have been added.
- Chapter 6. This chapter now covers applications of the definite integral, finding and using antiderivatives, and applications to probability. All but the first two sections are new to this book.
- Chapter 7. A new Focus on Theory section on the derivation of the regression line has been added.
- Chapter 8. This chapter covers differential equations, and is a new chapter. The focus is on qualitative behavior and applications and modeling. The applications covered include predator-prey problems and the spread of disease using systems of differential equations.

Supplementary Materials

The following supplementary materials are available.

- **Instructor's Manual with Sample Exam Questions** containing teaching tips, calculator programs, some overhead transparency masters and test questions arranged according to section.
- **Instructor's Solution Manual** with complete solutions to all problems.
- **Student's Solution Manual** with complete solutions to half the odd-numbered problems.

- **Instructor's Resource CD-ROM** which contains the Instructor's Manual, Instructor's Solutions Manual, as well as other valuable resources.
- **Student Study Guide** with study guides and supplementary materials, by Ansie Meiring of the University of Pretoria.

Getting Started Technology Manual Series:

- **Getting Started with Mathematica** by C-K. Cheung, G.E. Keough, Charles Landraitis, and Robert H. Gross of Boston College
- **Getting Started with Maple** by C-K. Cheung, G.E. Keough both of Boston College, and Michael May of St. Louis University
- **Getting Started with the TI-83/82 Graphing Calculator** by Carl Swenson of Seattle University
- **Getting Started with the TI-86/85 Graphing Calculator** by Carl Swenson of Seattle University
- **Getting Started with the TI-92/92 Plus Graphing Calculator** by Carl Swenson, Brian Hopkins both of Seattle University

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To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. At every stage, this book emphasizes the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of calculus in plain English. Success in using this book will depend on reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples.
- There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar–looking "worked out" examples. Success with the homework will come by grappling with the ideas of calculus.
- Many of the problems in the book are open-ended. This means that there is more than one correct approach and more than one correct solution. Sometimes, solving a problem relies on common sense ideas that are not stated in the problem explicitly but which you know from everyday life.
- This book assumes that you have access to a calculator or computer that can graph functions, find (approximate) roots of equations, and compute integrals numerically. There are many situations where you may not be able to find an exact solution to a problem, but can use a calculator or computer to get a reasonable approximation. An answer obtained this way is usually just as useful as an exact one. However, the problem does not always state that a calculator is required, so use your own judgement.

  If you mistrust technology, listen to this student, who started out the same way:

  > Using computers is strange, but surprisingly beneficial, and in my opinion is what leads to success in this class. I have difficulty visualizing graphs in my head, and this has always led to my downfall in calculus. With the assistance of the computers, that stress was no longer a factor, and I was able to concentrate on the concepts behind the shapes of the graphs, and since these became gradually more clear, I got increasingly better at picturing what the graphs should look like. It's the old story of not being able to get a job without previous experience, but not being able to get experience without a job. Relying on the computer to help me avoid graphing, I was tricked into focusing on what the graphs meant instead of how to make them look right, and what graphs symbolize is the fundamental basis of this class. By being able to see what I was trying to describe and learn from, I could understand a lot more about the concepts, because I could change the conditions and see the results. For the first time, I was able to see how everything works together . . . .

  That was a student at the University of Arizona who took calculus in Fall 1990, the first time we used some of the material in this text. She was terrified of calculus, got a C on her first test, but finished with an A for the course.
- This book attempts to give equal weight to three methods for describing functions: graphical (a picture), numerical (a table of values) and algebraic (a formula). Sometimes it's easier to translate a problem given in one form into another. For example, you might replace the graph of a parabola with its equation, or plot a table of values to see its behavior. It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups helpful. There are a great many problems which are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of one of the most important accomplishments of the millennium – calculus – as well as a real sense of the power of mathematics in the age of technology.

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CHAPTER ONE

FUNCTIONS AND CHANGE

Calculus enables us to study change. In this chapter we investigate total change and the average rate of change, and we use functions to represent how one quantity depends on another.

Functions are truly fundamental to mathematics. In everyday language we say, "The performance of the stock market is a function of consumer confidence" or "The patient's blood pressure is a function of the drugs prescribed." In each case, the word *function* expresses the idea that knowledge of one fact tells us another. In mathematics, the most important functions are those in which knowledge of one number tells us another number. If we know the length of the side of a square, its area is determined. If the circumference of a circle is known, its radius is determined.

Calculus starts with the study of functions. This chapter lays the foundation for calculus by surveying the behavior of some common functions. We also see ways of handling the graphs, tables, and formulas that represent these functions.

1.1 HOW DO WE MEASURE CHANGE?

Change is all around us. The temperature outside, the population of a town, the price of a stock, the size of a tumor, and the velocity of a baseball are all changing.

Change and Rate of Change

Kari is growing; her height each year on her birthday is given in Table 1.1. What is the change in Kari's height during the first four years of her life? We see that

$$\begin{pmatrix}\text{Change in height}\\ \text{between birth and age 4}\end{pmatrix} = 39 \text{ inches} - 19 \text{ inches} = 20 \text{ inches}.$$

What is the change in her height between age 4 and age 14?

$$\begin{pmatrix}\text{Change in height}\\ \text{between age 4 and age 14}\end{pmatrix} = 63 \text{ inches} - 39 \text{ inches} = 24 \text{ inches}.$$

TABLE 1.1 *Kari's height in inches*

Age (yrs)	Birth	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Height (in)	19	28	33	36	39	42	44	47	49	51	54	56	59	61	63

Was Kari growing faster during the first four years of her life or the following ten years? She grew 20 inches during the first 4 year period and she grew 24 inches during the following 10 years. However, the numbers 20 and 24 do not tell us when she was growing fastest. To answer this question, we use a *rate of change*. We calculate

$$\begin{pmatrix}\text{Average rate of change of height}\\ \text{between birth and age 4}\end{pmatrix} = \frac{\text{Change in height}}{\text{Change in age}} = \frac{39-19}{4-0} = \frac{20}{4} = 5 \text{ inches per year.}$$

$$\begin{pmatrix}\text{Average rate of change of height}\\ \text{between age 4 and age 14}\end{pmatrix} = \frac{\text{Change in height}}{\text{Change in age}} = \frac{63-39}{14-4} = \frac{24}{10} = 2.4 \text{ inches per year.}$$

Since Kari grew at an average rate of 5 inches per year up to age 4 and at an average rate of 2.4 inches per year between ages 4 and 14, Kari was growing faster during the first 4 years of her life.

Delta Notation

If we write y for Kari's height, we write Δy to represent the change in the value of y. (The symbol Δ is the Greek letter delta.) Similarly Δt stands for the change in the value of t. We write

$$\begin{pmatrix}\textbf{Change} \text{ in a quantity } y\\ \text{between time } a \text{ and time } b\end{pmatrix} = \begin{pmatrix}\text{Value of quantity}\\ \text{at time } b\end{pmatrix} - \begin{pmatrix}\text{Value of quantity}\\ \text{at time } a\end{pmatrix} = \Delta y$$

$$\begin{pmatrix}\textbf{Average rate of change} \text{ of a quantity } y\\ \text{between time } a \text{ and time } b\end{pmatrix} = \frac{\text{Change in quantity}}{\text{Change in time}} = \frac{\Delta y}{\Delta t}$$

The units of Δy are the units of y; the units of $\Delta y/\Delta t$ are the units of y divided by the units of t. The quantity $\Delta y/\Delta t$ is called a *difference quotient*.