FUNCTIONAL ANALYSIS

An Introduction for Physicists

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NINO BOCCARA

College of Liberal Arts and Sciences
University of Illinois at Chicago
Chicago, Illinois

and

Laboratoire de Mathématique
École de Physique et de Chimie
Paris, France

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Foreword

This book is intended for physicists who wish to familiarize themselves with the methods of functional analysis. These methods are extremely fruitful, but they cannot be properly used if one knows only unconnected recipes, ignoring, moreover, their conditions of applicability. That is not to say that the physicist has to be enslaved by a paralyzing rigor or indulge in a barren formalism. I have therefore sought to avoid the inflation of terminology, the introduction of concepts unnecessary to any further development, or proofs of too general a nature. Breaking away from the definition–theorem–proof style of exposition, I emphasize the practicality of the results, illustrating them with several examples. In order to assimilate a theorem, it is advisable to study the examples and to attempt to do the exercises, the solutions of which are provided, before tackling its proof.

This book is based on the course taught at the Ecole Supérieure de Physique et de Chimie in Paris. It is a modified and augmented second version of *Analyse Fonctionnelle: Une Introduction pour Physiciens*, Ellipses, Paris 1984. The prerequisite is a first course in analysis.

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Notations

The first digit refers to the number of the chapter in which the notation occurs for the first time, and the second refers to the section within the chapter. A stands for appendix, N for note, and P for problem.

$x \in A$: x is an element of the set A, 1.1
$\{x|P(x)\}$: the set of those x with property P, 1.1
$\forall$: for any, 1.1
$A \subset B$: set A is a subset of the set B, 1.1
$\cup, \cap$: union, intersection of sets, 1.1
1_A or χ_A: the characteristic function of the set A, 1.1
$\emptyset$: the empty set, 1.1
$\{a\}$: a set having a as unique element, 1.1
A^C: the complement of A, 1.1
$A - B$: the set of elements of A not in B, 1.1
$A \Delta B = (A - B) \cup (B - A)$: the symmetric difference of the sets A and B, 1.1
$\mathcal{P}(\mathrm{A})$: the set of all subsets of A, 1.1
$A \times B$: the cartesian product of the sets A and B, 1.1
(x, y): an ordered pair, 1.1
$f(x)$: the value of the mapping f at x, 1.1
$f(A)$: the image of the set A, 1.1
$f^{-1}(B)$: the preimage of the set B, 1.1
$x \mapsto f(x)$: the mapping (or the function) f, 1.1
$f^{-1}(y)$: the preimage of a one-element set $\{y\}$, 1.1
σ-algebra: a class of sets closed under countable unions and complements and containing the empty set, 1.1
$\mathcal{S}(\mathcal{E})$: the σ-algebra generated by $\mathcal{E}$, 1.1
$(X, \mathcal{A})$: the measurable space consisting of the space X and the σ-algebra $\mathcal{A}$, 1.1
$\mathbf{R}$: the set of real numbers, 1.1
$\overline{\mathbf{R}}$: the extended line, that is $\{-\infty\} \cup \mathbf{R} \cup \infty$, 1.1
$\mathbf{C}$: the set of complex numbers, 1.1
$]a, b[$, $[a, b]$: open, closed interval, 1.1
$[a, b[$, $]a, b]$: semiopen intervals, 1.1

$\sup_{x \in A} f(x)$, $\inf_{x \in A} f(x)$: supremum and infimum of f in A, 1.1
(f_n): a sequence of mappings, 1.1
$\lim_{n \to \infty} f_n$: the limit of the sequence (f_n), 1.1
$\limsup f_n$, $\liminf f_n$: the upper and lower limits of the sequence (f_n), 1.1
$f \circ g$: composed mapping of f and g, 1.1
f^+, f^-: positive, negative part of f, 1.1
$\Re$: real part, 1.1
$\Im$: imaginary part, 1.1
$x \notin A$: x is not an element of the set A, 1.2
$\mathbf{R}_+$: the set of nonnegative real numbers, 1.2
μ: positive measure, 1.2
$(X, \mathcal{A}, \mu)$: the measure space consisting of the space X, the σ-algebra $\mathcal{A}$ and the measure μ, 1.2
$\ell(I)$: the length of the interval I, 1.2
m^*: the outer measure, 1.2
m_*: the inner measure, 1.2
m: the Lebesgue measure, 1.2
$A + a = \{x + a | x \in A\}$, 1.2
xRy: equivalence relation, 1.2
$\mathbf{Q}$: the set of rational numbers, 1.2
$P \iff Q$: P is true if, and only if, Q is true, 1.2
$\mathbf{N}$: the set of positive integers, 1.2
$f(a - 0)$: the left-hand limit, that is, the limit of $f(x)$ when $x \to a$ and $x < a$, 1.2
$f(a + 0)$: the right-hand limit, that is, the limit of $f(x)$ when $x \to a$ and $x > a$, 1.2
a.e.: almost everywhere, that is, except on a negligible set, 1.2
$\int f \, d\mu$: integral of f with respect to μ, 1.3
$\int f \, dm$ or $\int f(x) \, dm(x)$ or $\int f(x) \, dx$: integral of f with respect to the Lebesgue measure, 1.3
$\mathbf{Z}$: the set of integers, 1.3
$\int_A f \, d\mu$: integral of f on the measurable set A, 1.3
$E(\xi)$: mathematical expectation of a random variable, 1.3
$V(\xi)$: variance of a random variable, 1.3
$|f|$: absolute value of the real or complex function f, 1.3
$\int_a^b f(x) \, dx$: Riemann integral of f on $[a, b]$, 1.3
$\mathcal{L}^1_\mu$: the vector space of integrable functions with respect to the measure μ, 1.3
J_n: Bessel function of the first kind of order n, 1.4

$\mu \otimes \nu$: product measure, 1.5
$(X \times Y, \mathcal{A} \otimes \mathcal{B})$: product of the measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, 1,5
$(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$: product of the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, 1,5
$E_x = \{y|(x,y) \in E\}$: x-section of the set E, 1.5
$E^y = \{x|(x,y) \in E\}$: y-section of the set E, 1.5
f_x: x-section of the function f, 1.5
f^y: y-section of the function f, 1.5
$\int f \, d\mu \otimes \nu = \int \int f(x,y) \, d\mu(x) \, d\nu(y)$: integral with respect to a product measure, 1.5
$\mathcal{M}$: monotone class, 1.5
$\omega_f(x)$: oscillation of f at x, 1.A
$P \Rightarrow Q$: P implies Q, 1.N2
$\|f\|$: norm of f, 1.N4
Γ: Euler Gamma function, 2.1
$\mathcal{L}^p_\mu(X)$: the vector space of measurable functions defined on X such that $\int |f|^p \, d\mu < \infty$, 2.2
$L^p_\mu(X)$: the normed vector space of measurable functions defined almost everywhere on X such that $\int |f|^p \, d\mu < \infty$, 2.2
$\tilde{f}$: the class of functions equivalent to f, 2.2
$\tilde{0}$: the class of functions equal to 0 a.e., 2.2
$\|f\|_p$: the L^p-norm of f, 2.2
$\mathcal{L}^\infty_\mu(X)$: the vector space of essentially bounded measurable functions defined on X, 2.2
ℓ^p: the space of sequences (x_n) such that $(|x_n|^p)$ is convergent, 2.2
$d(f,g)$: distance from f to g in a metric space, 2.2
ess sup f: the essential supremum of f, 2.2
$L^\infty_\mu(X)$: the normed vector space of essentially bounded measurable functions defined almost everywhere on X, 2.2
$d(x, A)$; distance between the point x and the set A, 2.3
$\overline{A}$: the closure of the set A, 2.3
$\mathring{A}$: the interior of the set A, 2.3
B: Euler Beta function, 2.3
$\hat{f}$: Fourier transform of the function f, 2.4
$\overline{x}$: the complex conjugate of x, 2.4
$u_n = O(n^k)$: $u_n n^k$ is bounded, 2.4
$L(f)$: Laplace transform of a function f, 2.5
θ: the Heaviside function, 2.5
$(x \mid y)$: scalar product of x and y, 3.1

$\|x\|$: norm of the vector x, 3.1
$L^2_\mu(X)$: the Hilbert space of square integrable functions defined almost everywhere on X, 3.2
$P_F(x)$: projection of the vector x on the subspace F, 3.2
E': the topological dual of the space E, 3.2
P_n: the Legendre polynomial of degree n, 3.3
$P_n^{(\alpha,\beta)}$: Jacobi polynomial, 3.3
$L_n^{(\alpha)}$: Laguerre polynomial, 3.3
H_n: the Hermite polynomial of degree n, 3.3
$G_n^{(p)}$: Gegenbauer polynomial, 3.3
T_n, U_n: Chebyshev polynomials, 3.3
ℓ^2: the Hilbert space of sequences (x_n) such that $(|x_n|^2)$ is convergent, 3.3
D_n: the Dirichlet kernel, 3.3
F_n: the Fejér kernel, 3.3
(E,d): metric space, 3.N2
C_c^∞: the space of infinitely differentiable functions having a bounded (compact) support, 4.1
$\mathcal{D} = C_c^\infty$: the space of test functions, 4.1
φ: a test function, 4.1
$T: \varphi \mapsto T(\varphi)$: the distribution T, 4.1
$\mathcal{D}'$: the space of Schwartz distributions, 4.1
L^1_{loc}: the space of locally integrable functions, 4.1
T_f: the regular distribution associated to the function f, 4.1
δ: the Dirac distribution, 4.1
Pv: Cauchy principal value, 4.2
Pf: pseudofunction, 4.2
Fp: Hadamard's finite part, 4.2
θ_s: characteristic function of the domain $S(x_1, x_2x_3) > 0$, 4.2
δ_s: the Dirac distribution on a surface, 4.2
A, **E**, **D**: tridimensional vectors, 4.2
n: unit tridimensional vector, 4.2
∇: nabla, 4.2
∇f: gradient of f, 4.2
$\nabla \cdot \mathbf{D}$: divergence of **D**, 4.2
$\nabla \times \mathbf{E}$: curl of **E**, 4.2
$S \otimes T$: the tensor product of the distributions S and T, 4.4
$S * T$: the convolution of the distributions S and T, 4.4
supp f: the support of the function f, 4.4
B: Euler Beta function, 4.4

S^1: a circle, 4.5
$c_k(T)$: Fourier coefficient of the distribution T, 4.5
$\widehat{T}$: Fourier transform of the distribution T, 4.6
$\check{T}$: inverse Fourier transform of the distribution T, 4.6
$L(T)$: Laplace transform of a distribution T, 4.7
γ: Euler constant, 4.7
CR: canonical regularization, 4.P4
X: the position operator, 5.1
P: the momentum operator, 5.1
D_A: the domain of a linear operator A, 5.1
R_A: the range of a linear operator A, 5.1
I: the identity operator, 5.1
A^{-1} the inverse of A, 5.1
$||A||$: the norm of the linear operator A, 5.2
$[A, B] = AB - BA$: the commutator of the operators A and B, 5.2
$\mathcal{B}(E)$: the Banach algebra of the linear operators on E, 5.2
$\widehat{A}$: extension of the operator A, 5.2
a_{mn}: matrix element of a bounded linear operator A, 5.2
P: projection operator, 5.2
A^+: the adjoint of the operator A, 5.2
F: Fourier or Fourier–Plancherel operator, 5.2
A^+, A: creation and annihilation operators, 5.2
G_A: the graph of the operator A, 5.2
$\overline{A}$: the closure of the operator A
$R_\lambda(A)$: the resolvent of the operator A, 5.5
$\rho(A)$: the resolvent set of the operator A, 5.5
$\sigma(A)$: the spectrum of the operator A, 5.5
$P\sigma(A)$: the point spectrum of the operator A, 5.5
$C\sigma(A)$: the continuous spectrum of the operator A, 5.5
$R\sigma(A)$: the residual spectrum of the operator A, 5.5
$E_1 \oplus E_2$: direct sum, 5.5

Contents

Chapter 1

Measure and Integration

Within the framework of Riemann's theory of integration, the validity of the formula

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx,$$

when (f_n) is a nonuniformly convergent sequence of Riemann-integrable functions, was extensively discussed at the end of the nineteenth century. The results obtained paved the way for Lebesgue's theory of integration.[1] This more general theory, better adapted for dealing with limit processes, allows the interchange of integral and limit in very general circumstances. The concept of measure plays a crucial role in Lebesgue's definition of the integral. Apart from Lebesgue's measure, which is a generalization of the notion of length, we shall consider different types of measures such as the Hausdorff measure. This particular measure is appropriate for Cantor-like sets found in many applications.

The most important results of this chapter are Lebesgue's dominated convergence theorem and Fubini's theorem. They are illustrated by many examples to show their wide applicability.

1. Measurable Functions

The theory of integration constructed in this chapter applies to measurable functions. Theorems 3 and 4 show that the class of measurable functions is very large; in physics, it is very unlikely that one would have to deal with nonmeasurable functions.

[1] On the origins and development of Lebesgue's theory, see T. Hawkins.

Definition 1. A class $\mathcal{A}$ of subsets of a set X is said to be a *σ-algebra* (sigma algebra) in X if
(1) X belongs to $\mathcal{A}$,
(2) any countable union of elements of $\mathcal{A}$ belongs to $\mathcal{A}$,
(3) the complement of any element of $\mathcal{A}$ is in $\mathcal{A}$.

From (1) and (3) it follows that the empty set $\emptyset$ is in $\mathcal{A}$; and from (2) and (3) it follows that any countable intersection of elements of $\mathcal{A}$ is in $\mathcal{A}$.

If A and B belong to $\mathcal{A}$, then $A - B = A \cap B^C$ and $A \Delta B = (A - B) \cup (B - A)$ are in $\mathcal{A}$.

The definition of a σ-algebra is similar to the definition of a topology (see Note 1). There is, however, an essential difference: the complement of an element of a σ-algebra belongs to the σ-algebra, but the the complement of an open set *is not* an open set.

Let $\mathcal{E}$ be a class of subsets of X; the set of all σ-algebras containing $\mathcal{E}$ is not empty since it contains, at least, $\mathcal{P}(X)$. The intersection of all these σ-algebras is a σ-algebra (see above), and, therefore, there exists a smallest σ-algebra containing $\mathcal{E}$. This σ-algebra is called the *σ-algebra generated by* $\mathcal{E}$; it is denoted $\mathcal{S}(\mathcal{E})$.

Theorem 1. *Let f be a mapping from X into Y, and let $\mathcal{B}$ be a σ-algebra in Y, then $\mathcal{A} = f^{-1}(\mathcal{B})$ is a σ-algebra in X.*

This result readily follows from the elementary formulas of set theory (see Note 2).

Definition 2. Let $\mathcal{A}$ be a σ-algebra in a space X, then the pair $(X, \mathcal{A})$ is called a *measurable space*, and the elements of $\mathcal{A}$ are called *measurable sets*.

Definition 3. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces; a mapping f from X into Y is said to be $\mathcal{A}$-$\mathcal{B}$ *measurable*[2] if, for any B in $\mathcal{B}$, $f^{-1}(B)$ belongs to $\mathcal{A}$.

In other words f is $\mathcal{A}$-$\mathcal{B}$ *measurable* if $f^{-1}(\mathcal{B}) \subset \mathcal{A}$.

Theorem 2. *Let f be a mapping from $(X, \mathcal{A})$ into $(Y, \mathcal{B})$, and let $\mathcal{E}$ be a class of subsets of Y generating the σ-algebra $\mathcal{B}$; then f is measurable if, and only if, $f^{-1}(\mathcal{E}) \subset \mathcal{A}$.*

Let us first prove the following lemma.

Lemma. *Let f be a mapping from X into Y, and let $\mathcal{E}$ be a class of subsets of Y; then $f^{-1}(\mathcal{S}(\mathcal{E})) = \mathcal{S}(f^{-1}(\mathcal{E}))$.*

---

[2] Or, simply, measurable if there is no ambiguity.

That is, the preimage of the σ-algebra generated by $\mathcal{E}$ is the σ-algebra generated by the preimage of $\mathcal{E}$.

From Theorem 1, it follows that $f^{-1}(\mathcal{S}(\mathcal{E}))$ is a σ-algebra. It contains $f^{-1}(\mathcal{E})$ and, therefore, $\mathcal{S}(f^{-1}(\mathcal{E}))$, which is the smallest σ-algebra containing $f^{-1}(\mathcal{E})$. Thus

$$f^{-1}(\mathcal{S}(\mathcal{E})) \supset \mathcal{S}(f^{-1}(\mathcal{E})).$$

But (see Note 2), the set

$$\{B | B \subset Y, f^{-1}(B) \in \mathcal{S}(f^{-1}(\mathcal{E}))\}$$

is a σ-algebra $\mathcal{S}'$ which contains $\mathcal{E}$ and, therefore, $\mathcal{S}(\mathcal{E})$. Thus

$$f^{-1}(\mathcal{S}(\mathcal{E})) \subset f^{-1}(\mathcal{S}') \subset \mathcal{S}(f^{-1}(\mathcal{E})),$$

which completes the proof.

From this lemma, it follows that

$$f^{-1}(\mathcal{B}) = f^{-1}(\mathcal{S}(\mathcal{E})) = \mathcal{S}(f^{-1}(\mathcal{E})).$$

Hence, $f^{-1}(\mathcal{B}) \subset \mathcal{A}$ if, and only if, $f^{-1}(\mathcal{E}) \subset \mathcal{A}$.

Corollary 1. *Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$, $(Z, \mathcal{C})$ be three measurable spaces, and let f and g be, respectively, two measurable functions from $(X, \mathcal{A})$ into $(Y, \mathcal{B})$ and from $(Y, \mathcal{B})$ into $(Z, \mathcal{C})$; then $g \circ f$ is a measurable function from $(X, \mathcal{A})$ into $(Z, \mathcal{C})$.*

This is obvious.

Definition 4. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces; then $(X \times Y, \mathcal{A} \otimes \mathcal{B})$, where $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra generated by $\{A \times B | A \in \mathcal{A}, B \in \mathcal{B}\}$, is called the *product of the measurable spaces* $(X, \mathcal{A})$ and $(Y, \mathcal{B})$.

Corollary 2. *Let $(X, \mathcal{A})$, $(Y_1, \mathcal{B}_1)$ and $(Y_2, \mathcal{B}_2)$ be three measurable spaces, and let f be a mapping from X into $Y_1 \times Y_2$; f is measurable if, and only if, the mapping $p_1 \circ f$ and $p_2 \circ f$, where p_1 and p_2 are, respectively, the canonical projections of $Y_1 \times Y_2$ on Y_1 and Y_2, are measurable.*

For any ordered pair (y_1, y_2) in $Y_1 \times Y_2$, we have

$$p_1 : (y_1, y_2) \mapsto y_1 \qquad \text{and} \qquad p_2 : (y_1, y_2) \mapsto y_2.$$

Since $p_1(\mathcal{B}_1)$ and $p_2(\mathcal{B}_2)$ are both contained in $\mathcal{B}_1 \otimes \mathcal{B}_2$ (Definition 4), p_1 and p_2 are measurable. f being measurable, as a consequence of Corollary 1, $p_1 \circ f$ and $p_2 \circ f$ are, therefore, measurable.

Conversely, if $p_1 \circ f$ and $p_2 \circ f$ are measurable,

$$(\forall B_1 \in \mathcal{B}_1) \quad (\forall B_2 \in \mathcal{B}_2) \qquad f^{-1}(B_1 \times B_2) \in \mathcal{A}$$

since

$$f^{-1}(B_1 \times B_2) = (p_1 \circ f)^{-1}(B_1) \cap (p_2 \circ f)^{-1}(B_2);$$

and, therefore, f is measurable (Theorem 2).

Example 1. *Borel measurable mappings.* Let X be a topological space, and let $\mathcal{B}$ be the σ-algebra generated by the open sets of X. The elements of $\mathcal{B}$ are called the *Borel sets* of X. The closed sets of X are Borel sets, and so are all countable unions of closed sets and all countable intersections of open sets.

If $X = \mathbf{R}$, the σ-algebra $\mathcal{B}$ is generated by the set $\{]a, \infty[\,|\, a \in \mathbf{R}\}$. The intervals $[a, \infty[$, $]-\infty, a[$, $]a, b[$, and $[a, b]$ belong to $\mathcal{B}$ since

$$[a, \infty[= \cap_{n=1}^{\infty}]a - \frac{1}{n}, \infty[$$

$$]-\infty, a[= [a, \infty[^C$$

$$]a, b[=]-\infty, b[\cap]a, \infty[$$

$$[a, b] = [a, \infty[-]b, \infty[.$$

Any topological space can be regarded as a measure space with the Borel sets playing the role of the measurable sets. Let X and Y be two topological spaces; a measurable mapping $f: X \to Y$ is said to be *Borel measurable.* For instance, any continuous mapping $f: X \to Y$ is Borel measurable, since, for any open set Ω of Y, $f^{-1}(\Omega)$ is a Borel set of X (Theorem 2).

In what follows we shall deal, almost exclusively, with mappings from X into $\mathbf{R}$ (or $\mathbf{C}$), where X is any space. These particular mappings will be called *functions*, and to prove that they are measurable, we shall make use of the following criterion.

Criterion. *Let f be a function from X into $\mathbf{R}$; f is measurable if, and only if, for any real a, the set $\{x|f(x) > a\}$ is measurable.*

This result follows from Theorem 2 and from the fact that the σ-algebra of the Borel sets of $\mathbf{R}$ is generated by the set $\{]a, \infty[\,|\, a \in \mathbf{R}\}$.

Remark. Equivalent criteria could have been established by considering the sets $\{x|f(x) \geq a\}$, $\{x|f(x) < a\}$, or $\{x|f(x) \leq a\}$.

Example 2. If f is measurable, then the set $\{x|f(x) = a\}$ is a Borel set. If a is finite, this result follows from the fact that

$$\{x|f(x) = a\} = \{x|f(x) \geq a\} \cap \{x|f(x) \leq a\}.$$

If $a = \infty$, the result follows from

$$\{x|f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x|f(x) > n\}.$$

If $a = -\infty$, the proof is similar.

Example 3. The characteristic function 1_A of a set A is measurable if, and only if, the set A is measurable, since

$$\{x|1_A(x) > a\} = \begin{cases} \mathbf{R}, & \text{if } a < 0; \\ A, & \text{if } 0 < a < 1; \\ \emptyset, & \text{if } a > 1. \end{cases}$$

As a consequence of Corollary 2, a function f of X in $\mathbf{C}$ is measurable if, and only if, $\Re f$ and $\Im f$ are measurable.

Theorem 3. *Let f and g be two measurable functions from X into $\mathbf{R}$ (or $\mathbf{C}$).*

(1) *For any positive number α, $|f|^\alpha$ is measurable.*

(2) *If there is no $x \in X$ such that $f(x) = 0$, $1/f$ is measurable.*

(3) *$f + g$ and fg are measurable.*

(1) $|f|^\alpha$ is the composed mapping of f and $z \mapsto |z|^\alpha$. This last mapping is continuous and, therefore, Borel measurable. The result is then a consequence of Corollary 1.

(2) The proof is similar to the preceding one, since the mapping $z \mapsto 1/z$ from $\mathbf{R}-\{0\}$ (or from $\mathbf{C}-\{0\}$) into $\mathbf{R}$ is continuous.

(3) $f+g$ (resp. fg) is the composed mapping of $x \mapsto (f(x), g(x))$ from X into $\mathbf{R}^2$ (or $\mathbf{C}^2$), which is measurable (Corollary 2), and of $(z_1, z_2) \mapsto z_1 + z_2$ (resp. $z_1 z_2$), which is Borel measurable (because continuous). The result is then a consequence of Corollary 1.

Example 4. Let f be a function whose range is $\{a_1, a_2, \ldots, a_n\}$, where a_i ($i = 1, 2, \ldots, n$) are n distinct finite real numbers. If $A_i = \{x|f(x) = a_i\}$, then

$$f = \sum_{i=1}^{n} a_i 1_{A_i}.$$

If the sets A_i ($i = 1, 2, \ldots, n$) are measurable, the characteristic functions 1_{A_i} ($i = 1, 2, \ldots, n$) are measurable (see Example 2), and Theorem 3(3) shows that f is measurable. Finite linear combinations of characteristic functions of measurable sets will play a very important role in the construction of Lebesgue's theory (see Definition 4 and Theorem 5).