CALCULUS ∪ Analytic Geometry ∪ Vectors

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Preface

The physical sciences have deep roots in mathematics, and mathematics is now permeating such traditionally qualitative subjects as business, economics, linguistics, medicine, and psychology. These applications, moreover, increasingly depend upon basic principles rather than handbook recipes. Even so, an early display of rigor may be viewed as exalting the beauty of mathematics over its utility. Until the need for the epsilon-delta method is felt, there is danger of stifling interest by force-feeding this powerful technique too soon. This book, therefore, begins on a descriptive level and proceeds in a gradually tightening atmosphere of rigor consistent with the student's development. After an intuitive discussion, for example, limits are defined, but how a delta depends on an epsilon is subtly worked in later. Limit theorems are stated in Chapter 2 and the simplest proofs given, but the other proofs are more likely to be honored in their setting of Chapter 13, after repeated evidence that intuition cannot always be relied upon even in applications.

Geometric vectors are natural models for directed quantities, and a novice forced to jump this visual aid may be lost in a morass of ordered triples, dots, and crosses. With the background of vectors in this book, however, later progress should be eased when the student sees that the founders of linear spaces transcribed the same pictures into symbols. The vector notion is also the catalyst unifying rectangular and polar coordinates, rotation of axes, parametric equations, curvilinear motion, and complex numbers. Isaac Newton (1642–1727) leaned heavily on vector concepts in formulating his ideas leading to organized calculus. The present interest in artificial satellites, incidentally, makes the spark of Newton's genius shine brighter than ever, as shown in the section on space travel. Even though vectors appear in the first chapter and repeatedly thereafter, they are considered to be tools, as are algebra and set notation. In particular, currently fashionable Boolean algebra is not included since the needs of calculus and analytic geometry are satisfied, and greatly aided, by a bare minimum of unions and intersections.

There is ample evidence that upper and lower Darboux sums squeeze in too much, and the step-function ladder reaches too high, for initial exposure of most students to definite integrals. Also, an attenuated version of measure theoretic methods using Jordan content fosters a delusion, hard to counteract, that definite integrals are primarily for finding areas. The definite integral is defined in this book as the limit, when it exists, of finite sums, its role in defining continuous quantities as extensions of discrete ones is featured, and it is used early and frequently. The more sophisticated methods are excellent for a later course, but prior to a sharp awareness of the potency of definite integrals, a detailed development would consume time and energy out of proportion with the benefits.

So many persons have contributed to this book that I dare not name any for fear of missing some. I have, however, personally thanked all I can remember.

John F. Randolph

Contents

Chapter 1

Vectors

1-1. Geometric Vectors and Vector Algebra

Vector analysis originated in the desire of nineteenth-century physicists and mathematicians to deal with such quantities as forces and velocities whose properties are not specified in terms of numbers alone. Directed line segments furnished a natural geometrical model for such quantities. Both the directions and magnitudes of forces F_1 and F_2 applied at a point of a body were modeled by directed line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ with the same initial point O. The single force F producing the same effect on the body is represented by the directed diagonal $\overrightarrow{OR}$ of a parallelogram as in Fig. 1-1.1. Thus, the (geometric) parallelogram law for adding vectors emerged to give

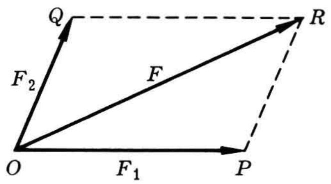

Fig. 1-1.1

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}.$$

Physical concepts may then be interpreted in terms of geometrical properties.

The physical counterpart of the geometrical operation of moving $\overrightarrow{OQ}$ to $\overrightarrow{PR}$ would apply the force F_2 at a different point of the body and produce a different effect on the body. The same directed line segment $\overrightarrow{OR}$ is obtained, however, by the triangle law illustrated in Fig. 1-1.2. The algebraic expression

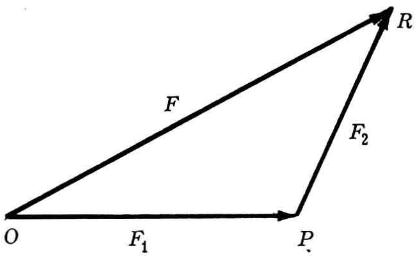

Fig. 1-1.2

$$\overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

of the triangle law then furnishes a "cancellation" technique not evident in $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$. Hence, the concept arose of a vector being free to move parallel to its original position. Although $\overrightarrow{OQ}$ is geometrically distinct from $\overrightarrow{PR}$, the substitution of $\overrightarrow{PR}$ for $\overrightarrow{OQ}$ in

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

has a meaningful interpretation, and thus algebraically $\overrightarrow{PR} = \overrightarrow{OQ}$.

The concepts of bound and free vectors then developed. A bound vector $\vec{v}$ having O as its initial point is merely a directed line segment $\overrightarrow{OP}$. A bound vector is completely determined by specifying its initial and terminal points. The length (in terms of some preassigned unit) of the line segment joining O and P is called the **norm** of $\vec{v}$ and is denoted by $|\vec{v}| = |\overrightarrow{OP}|$. A vector $\vec{v}$ having $|\vec{v}| = 1$ is called a **unit vector**.

In contrast, a free vector $\vec{v}$ was conceived as a whole collection of directed line segments, any one of which is a parallel translation of any other. Thus, if $\overrightarrow{PR}$ and $\overrightarrow{OQ}$ are specific bound vectors of a free vector $\vec{v}$, then $\overrightarrow{PR}$ and $\overrightarrow{OQ}$ have the same norm and the same direction but different initial points. Even though $\overrightarrow{PR}$ and $\overrightarrow{OQ}$ are geometrically distinct, either may be substituted for the other in the algebra associated with free vectors, as developed below. Hence, the free vector $\vec{v}$ is represented by either $\overrightarrow{PR}$ or $\overrightarrow{OQ}$. It is customary to set

$$\vec{v} = \overrightarrow{PR} \text{ or } \vec{v} = \overrightarrow{OQ}$$

and to interpret these as specifying the norm and direction common to all those directed line segments (Fig. 1-1.3) constituting the free vector $\vec{v}$.

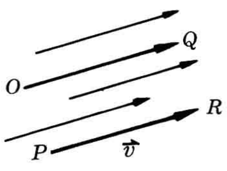

Fig. 1-1.3

The word "vector" without an adjective will generally mean "free vector." Whenever a specific bound vector is used to represent a free vector, this will be indicated by context. Thus, the norm $|\vec{v}|$ of a vector $\vec{v}$ is the norm of any (and all) of the representatives of $\vec{v}$.

Given two vectors $\vec{u}$ and $\vec{v}$, take any representative $\overrightarrow{OP}$ of $\vec{u}$ and represent $\vec{v}$ by $\overrightarrow{PQ}$. Then $\overrightarrow{OQ}$ represents a vector called the **sum** (or resultant) of $\vec{u}$ and $\vec{v}$ and denoted by $\vec{u} + \vec{v}$. But $\vec{v}$ could have been represented first by $\overrightarrow{OP_1}$ and then P_1 used as the initial point of a representative $\overrightarrow{P_1Q_1}$ of $\vec{u}$ to obtain $\vec{v} + \vec{u}$ represented by $\overrightarrow{OQ_1}$. By similar triangles, Q_1 and Q are at the same point, so that

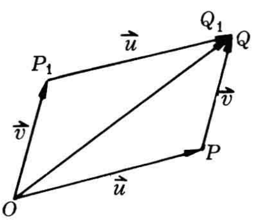

Fig. 1-1.4

(1) $$\vec{u} + \vec{v} = \vec{v} + \vec{u}.$$

With $\vec{w}$ a third vector, Fig. 1-1.5a gives $\overrightarrow{OR}$ as a representative of $(\vec{u} + \vec{v}) + \vec{w}$. The superposition of Fig. 1-1.5b upon Fig. 1-1.5a shows that $\overrightarrow{OR}$ also represents $\vec{u} + (\vec{v} + \vec{w})$, so that

(2) $$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}).$$

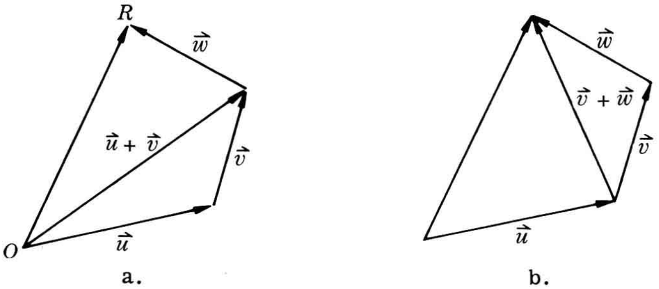

Fig. 1-1.5

Hence, in forming the sum of three vectors it is not necessary to associate two of them before adding the third, and thus the sum may be written

$$\vec{u} + \vec{v} + \vec{w}.$$

By reversing the direction of each representative of a vector $\vec{v}$, an associated vector denoted by $-\vec{v}$ is obtained. The actual distance between the endpoints of any representative of $-\vec{v}$ is the same as for $\vec{v}$, so that

$$|-\vec{v}| = |\vec{v}|.$$

Then, with $\vec{u}$ a vector, the subtraction of $\vec{v}$ from $\vec{u}$ is defined by

(3) $$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}).$$

Hence, if the representations $\vec{u} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{OQ}$ are used, the directed segment $\overrightarrow{QP}$ represents $\vec{u} - \vec{v}$, as shown in Fig. 1-1.6. Algebraically, $-\vec{v} = \overrightarrow{QO}$ and

$$\vec{u} + (-\vec{v}) = \overrightarrow{OP} + \overrightarrow{QO} = \overrightarrow{QO} + \overrightarrow{OP} = \overrightarrow{QP}.$$

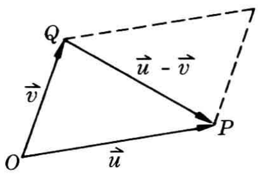

Fig. 1-1.6

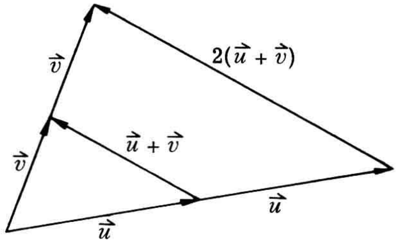

Fig. 1-1.7

If each representative of a vector $\vec{v}$ is doubled in length, without change of direction, the result is a vector denoted by $2\vec{v}$. Thus,

$$2\vec{u} + 2\vec{v} = 2(\vec{u} + \vec{v})$$

follows, since the lengths of corresponding sides of two similar triangles are in the same ratio (Fig. 1-1.7).

Whenever numbers and vectors are used together, the numbers are referred to as **scalars**. Thus, each scalar a has an absolute value denoted by $|a|$, where, by definition,

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0. \end{cases}$$

For example, $|-2| = |2| = 2$ and $|x| = 0$ if and only if $x = 0$.

With a a nonzero scalar and $\vec{v}$ a vector, then a vector denoted by $a\vec{v}$ is obtained by:

(a) multiplying the length of each representative of $\vec{v}$ by $|a|$ and then
(b) reversing the direction if and only if a is negative. In particular, $-\vec{v} = (-1)\vec{v}$ and $-(2\vec{v}) = (-2)\vec{v} = 2(-\vec{v})$.

It is now convenient to extend the vector notion to include the so-called **null** (or zero) vector $\vec{0}$ as having zero norm but with no direction assigned. The null vector is not represented geometrically, but is used algebraically with any vector $\vec{v}$ as:

(4) $$\vec{v} + \vec{0} = \vec{v},$$

(5) $$\vec{v} - \vec{v} = \vec{0}, \text{ and}$$

(6) $$a\vec{v} = \vec{0} \text{ if either } a = 0 \text{ or } \vec{v} = \vec{0}.$$

The following laws then hold even if null vectors or zero scalars appear:

(7) $$a\vec{v} + b\vec{v} = (a + b)\vec{v},$$

(8) $$a\vec{u} + a\vec{v} = a(\vec{u} + \vec{v}),$$

(9) $$\text{if } a\vec{v} = b\vec{v} \text{ and } \vec{v} \neq \vec{0}, \text{ then } a = b, \text{ and}$$

(10) $$\text{if } a\vec{u} = a\vec{v} \text{ and } a \neq 0, \text{ then } \vec{u} = \vec{v}.$$

1-2. Basic Unit Vectors

Let O be a given point. A standard notation is $\vec{i}$ and $\vec{j}$ for two vectors perpendicular to one another, each one unit long and each with initial end at O. The vectors are called **basic unit vectors** of the plane in which they lie. It is usual to have $\vec{i}$ horizontal pointing right and $\vec{j}$ vertical pointing up (toward the top of the page).

Given a vector $\vec{v}$ in the plane of $\vec{i}$ and $\vec{j}$ and also with initial end at O, there are scalars x and y such that

(1) $$\vec{v} = x\vec{i} + y\vec{j}.$$

To see this, let $\overrightarrow{OP_1}$ be the vector projection of $\vec{v}$ on the line containing $\vec{i}$. Then $\overrightarrow{OP_1}$ is a multiple of $\vec{i}$; that is, there is a scalar x such that $\overrightarrow{OP_1} = x\vec{i}$. This scalar x may be positive, negative, or zero. In the same way, a vector $\overrightarrow{OP_2} = y\vec{j}$ is determined by projecting $\vec{v}$ onto the line containing $\vec{j}$. Then

$$\vec{v} = \overrightarrow{OP_1} + \overrightarrow{OP_2} = x\vec{i} + y\vec{j}.$$

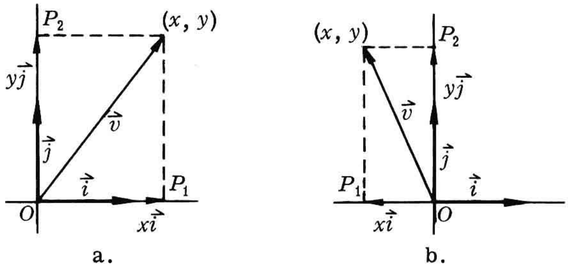

Fig. 1-2.1

The terminal end of $\vec{v}$ such that (1) holds is labeled (x, y). Hence, a one-to-one correspondence is established between vectors $\vec{v}$ with initial ends at O and ordered pairs (x, y) of scalars. It is then unambiguous to speak of "the vector (x, y)" instead of "the vector $x\vec{i} + y\vec{j}$ with initial end at O." For example, $(2, -1) + (1, 5) = (3, 4)$ is another way of indicating the vector addition

$$(2\vec{i} - \vec{j}) + (\vec{i} + 5\vec{j}) = (2 + 1)\vec{i} + (-1 + 5)\vec{j} = 3\vec{i} + 4\vec{j}.$$

As defined earlier, the actual length of a vector $\vec{v}$, denoted by $|\vec{v}|$, is called the **norm** (or absolute value) of $\vec{v}$. With $\vec{v} = x\vec{i} + y\vec{j}$, then, by the Pythagorean theorem,

(2) $$|\vec{v}| = \sqrt{x^2 + y^2}.$$

Hence, if $\vec{v} \neq \vec{0}$, then $|\vec{v}| \neq 0$, or, equivalently, if $|\vec{v}| = 0$, then $\vec{v} = \vec{0}$.

With basic unit vectors $\vec{i}$ and $\vec{j}$ selected, then a label (x, y) for each point of the plane is determined. It is then said that a **plane coordinate system** has been established. "The point labeled (x, y)" is shortened to "the point (x, y)." Also, x is called the **abscissa** and y the **ordinate** of the point. The whole line containing $\vec{i}$ is called the **axis of abscissas** (or x-axis), while the half of this line with endpoint O and containing $\vec{i}$ is called the **positive *x*-axis,** with the other half called the **negative *x*-axis**. Similar definitions are given for the **axis of ordinates**. The point O is called the **origin.**

Saying "the vector (x, y)" or "the point (x, y)" establishes the context as primarily concerned with vectors or coordinates, respectively. The interplay between vectors and coordinate geometry is an aid to both.

With points (x_1, y_1) and (x_2, y_2) given, the vectors

$$\vec{v}_1 = x_1\vec{i} + y_1\vec{j} \text{ and } \vec{v}_2 = x_2\vec{i} + y_2\vec{j},$$

with initial ends at the origin, terminate at the points (x_1, y_1) and (x_2, y_2). The vector difference

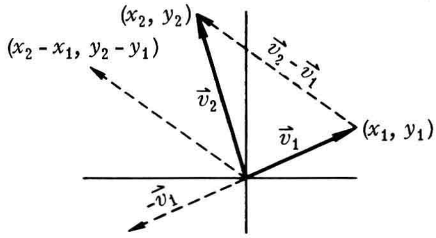

Fig. 1-2.2

$$\vec{v}_2 - \vec{v}_1 = (x_2\vec{i} + y_2\vec{j}) - (x_1\vec{i} + y_1\vec{j}) = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$$

may be pictured as the vector from the point (x_1, y_1) to the point (x_2, y_2) or else as the vector from the origin to the point $(x_2 - x_1, y_2 - y_1)$. Hence, by (2), the norm of this vector,

$$|v_2 - v_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

may be visualized as the distance between the points (x_1, y_1) and (x_2, y_2).

The vector from the origin to the point two-fifths of the way from the point (x_1, y_1) to (x_2, y_2) is

$$\begin{aligned}\vec{v} &= \vec{v}_1 + \tfrac{2}{5}(\vec{v}_2 - \vec{v}_1) = \tfrac{3}{5}\vec{v}_1 + \tfrac{2}{5}\vec{v}_2\\ &= \tfrac{3}{5}(x_1\vec{i} + y_1\vec{j}) + \tfrac{2}{5}(x_2\vec{i} + y_2\vec{j})\\ &= \frac{3x_1 + 2x_2}{5}\vec{i} + \frac{3y_1 + 2y_2}{5}\vec{j}.\end{aligned}$$

The terminal end of this vector is therefore the point

$$\left(\frac{3x_1 + 2x_2}{5}, \frac{3y_1 + 2y_2}{5}\right).$$

By a similar method, show: *The line segment joining the points (x_1, y_1) and (x_2, y_2) has midpoint*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problem 1. Show that each of the vectors has unit norm:

a. $\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$.
c. $-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.
e. $\frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}$.
b. $\frac{\sqrt{3}}{2}\vec{i} - \frac{1}{2}\vec{j}$.
d. $-\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$.
f. $0.8\vec{i} - 0.6\vec{j}$.

Problem 2. Draw the vector and find its norm:

a. $3\vec{i} + 4\vec{j}$.
c. $3\vec{i} - 3\vec{j}$.
e. $-\sqrt{2}\,\vec{i} + 0\vec{j}$.
b. $5\vec{i} - 12\vec{j}$.
d. $\vec{i} + \vec{j}$.
f. $0\vec{i} - 5\vec{j}$.