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Sergey V. Matveev

Lectures on Algebraic Topology

European Mathematical Society

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Translated by Ekaterina Pervova

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Author:

Sergey V. Matveev
Department of Computer Topology and Algebra
Chelyabinsk State University
ul. Br. Kashirinykh 129
454021 Chelyabinsk
Russia

Translator:

Ekaterina Pervova
Department of Computer Topology and Algebra
Chelyabinsk State University
ul. Br. Kashirinykh 129
454021 Chelyabinsk
Russia

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Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
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Phone: +41 (0)1 632 34 36
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Preface

Algebraic topology is the study of geometric objects via algebraic methods. Familiarity with its main ideas and methods is quite useful for all undergraduate and graduate students who specialize in any of the many branches of mathematics and physics that have connections to topology, differential geometry, algebra, mathematical analysis, or differential equations. In selecting the content of this book and in writing it the author aspired to reach the following goals:

- to cover those ideas and results that form the backbone of algebraic topology and are sufficient to provide a beautiful, intuitively clear, and logically complete exposition;
- to make the book self-contained, while keeping it reasonably short;
- to make the exposition logically coherent, well-illustrated, and mathematically rigorous, at the same time preserving all the advantages of an informal and lively presentation;
- to structure the text and supplement it with exercises and solutions in such a way that the book becomes a ready-to-use tool for both teachers and students of the subject, as well as a convenient instrument for independent study.

A special attention was devoted to providing explicit algorithms for calculating the homology groups and for manipulating fundamental groups. These subjects are often missing from other books on algebraic topology.

The present book is a revised and slightly extended version of the Russian original publication.

Contents

1

Elements of homology theory

1.1 Categories and functors

One of the main parts of algebraic topology is homology theory, which is a functor from the category of topological spaces to the category of sequences of Abelian groups. Therefore we begin by introducing the notions of a category and a functor.

In order to define a category, we proceed as follows:

1. We specify a certain class of *objects*. Objects may be of any nature.
2. For every ordered pair A, B of objects we specify a set of *morphisms* $[A, B]$ of object A to object B.
3. For every ordered triple A, B, C of objects we indicate a rule assigning to each pair of morphisms $f \in [A, B], g \in [B, C]$ a third morphism, which belongs to $[A, C]$, is called a *composition* of morphisms f, g, and is denoted by gf. In other words, we define a *composition map* $[A, B] \times [B, C] \to [A, C]$.

Definition. The class of objects, the sets of morphisms, and the composition maps thus specified form a *category* if the following axioms hold:

I. Composition of morphisms must be associative, *i.e.* for all triples of morphisms $f \in [A, B], g \in [B, C], h \in [C, D]$ we must have an equality $(hg)f = h(gf)$.

II. For any object B there must be a morphism $\mathrm{Id}_B \in [B, B]$ such that for any two morphisms $f \in [A, B]$, $g \in [B, C]$ the following equalities hold: $\mathrm{Id}_B\, f = f$ and $g\, \mathrm{Id}_B = g$.

Such situations (classes of objects related by morphisms satisfying axioms I, II; *i.e.* categories) arise naturally in many different areas of mathematics. Already at this general level it is possible to give definitions and prove meaningful theorems, which, by virtue of their generality, enjoy a remarkably wide applicability. We restrict ourselves to a very brief introduction into category theory. A more detailed exposition can be found, for instance, in [2].

Examples of categories

1. The category of all sets and their maps. The objects of this category are all sets, the morphisms are all possible maps between them.

2. The category of groups and homomorphisms. The objects are groups, and the morphisms are their homomorphisms.
3. The category of Abelian groups and their homomorphisms. The objects of this category are Abelian groups, the morphisms are homomorphisms between them.

It is easy to find other examples of a similar kind: the category of finitely generated groups, the category of rings, and others.

4. The category of topological spaces and continuous maps. The objects are all topological spaces, and the morphisms are their continuous maps.

In all of the above examples the objects are sets, perhaps with additional structures, and the morphisms are maps of sets. However, there exist categories of other types.

5. The category of topological spaces and classes of homotopic maps. The objects of this category are topological spaces, the morphisms are classes of homotopic maps (see definition on page 15). Notice that in this category morphisms are not maps themselves but rather classes of homotopic maps.

Although morphisms do not have to be actual maps, it is quite convenient to denote them in the same way as maps: instead of $f \in [A, B]$, we write $f : A \to B$.

Definition. Objects X, Y of a category G are called *isomorphic* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{Id}_Y$ and $gf = \mathrm{Id}_X$, where $\mathrm{Id}_X, \mathrm{Id}_Y$ are the identity morphisms of objects X and Y. The morphisms f, g are called *isomorphisms*.

Example. Which sets are isomorphic in the category of all sets? It is easy to see that those are exactly sets of equal cardinality, since an isomorphism in this category is nothing other than a bijection.

Example. An isomorphism in the category of topological spaces is any homeomorphism; in the category of groups, a group isomorphism. An isomorphism in the category of topological spaces and homotopic maps is called a *homotopy equivalence*. It is also worthwhile to mention an isomorphism in the category of smooth manifolds and smooth maps. That is a *diffeomorphism*.

The usefulness of the notion of a category can be seen already from these examples: a single definition stated in terms of category theory, can replace many corresponding definitions in specific categories. A similar fact is true for theorems as well: if a theorem is proven in categorical terms, then it automatically holds in all specific categories. This observation yields a promising method of obtaining new results.

Let $\mathsf{G}_1, \mathsf{G}_2$ be two categories. Suppose that to each object X of the former category we assign an object of the latter category. Let us denote it by $F(X)$. Assume also that to each morphism $f : X \to Y$ of G_1 we assign a morphism $f_* : F(X) \to F(Y)$ of G_2. Such an assignment is called a *covariant functor* from the category G_1 to the category G_2 if the following axioms hold:

1. If f is an identity morphism, then f_* is also an identity morphism.

2. If a composition fg is well defined, then $(fg)_* = f_* g_*$.

A *contravariant functor* F from a category G_1 to a category G_2 differs from a covariant one in that to each morphism $f : X \to Y$ of the category G_1 we assign a morphism $f^* : F(Y) \to F(X)$ of the category G_2, *i.e.* one that acts in the opposite direction. The axioms are of course changed in the natural way. In particular, the equality $(fg)_* = f_* g_*$ is replaced by the equality $(fg)^* = g^* f^*$.

Exercise 1. Give examples of covariant and contravariant functors.

Theorem 1. *Let $F : \mathsf{G}_1 \to \mathsf{G}_2$ be a functor from a category G_1 to a category G_2. Suppose that two objects X, Y of the category G_1 are isomorphic. Then the objects $F(X)$, $F(Y)$ of the category G_2 are also isomorphic. Equivalently, if the objects $F(X)$, $F(Y)$ of the category G_2 are not isomorphic then neither are the objects X, Y.*

Proof. We limit ourselves to considering a covariant functor. Let $f : X \to Y$ and $g : Y \to X$ be some isomorphisms between X and Y such that $fg = \mathrm{Id}_Y$ and $gf = \mathrm{Id}_X$. Then it follows immediately from the definition of a functor that $f_* g_* = \mathrm{Id}_{F(Y)}$ and $g_* f_* = \mathrm{Id}_{F(X)}$, which ensures that objects $F(X)$ and $F(Y)$ are isomorphic as well. □

Theorem 1 is of fundamental importance. Here is a standard way of applying it: suppose that we want to find out whether some given topological spaces X and Y are distinct. Take a functor from the category of topological spaces to another category, for instance, a category of groups, and compare the objects $F(X)$ and $F(Y)$. If they are distinct, then X and Y are distinct as well. In case $F(X)$ and $F(Y)$ coincide, nothing can be said about X and Y. This remark explains the significance of homology theory, which is a functor from topology to algebra.

Exercise 2. Applying Theorem 1, show that the cyclic groups Z_4 and Z_5 are not isomorphic.

Thus, with the help of a functor $F : \mathsf{G}_1 \to \mathsf{G}_2$, the problem of distinguishing objects in the category G_1 is replaced by a similar problem of distinguishing objects in the category G_2. The meaning of the replacement is that in G_2 this problem may be easier. It should be noted that when we pass from the category G_1 to G_2, a part of the information about the objects of G_1 is usually lost.

A careful consideration of the above arguments shows that a "nice" functor should possess the following properties:

1. It should be easily computable, *i.e.* the determination of the object $F(X)$ for a given space X should not pose difficulties of fundamental nature.

2. There should be a simple way of distinguishing objects $F(X)$ and $F(Y)$.

3. The transition from an object X to $F(X)$ should not lose too much information.

Homology functors from the category of topological spaces to the category of groups meet these requirements to a significant extent. These aspects should be given special attention when studying homology theory.

In fact, a homology functor assigns to a topological space not a single group but rather a whole sequence of Abelian groups, *i.e.* it is a functor to the category of sequences of Abelian groups. Calculating the homology groups of an arbitrary space may prove to be unpredictably difficult, therefore, as a rule, they are studied for a class of spaces that are not too complicated. We take the category of simplicial complexes as the domain of our homology functor and only briefly mention more general homology theories.

1.2 Some geometric properties of $\mathbb{R}^N$

Recall that a basis in the Euclidean space $\mathbb{R}^N$ is an ordered collection of N linearly independent vectors.

Definition. Two bases of $\mathbb{R}^N$ are called *equivalent* if the determinant of any change of coordinates matrix between them is positive.

Exercise 3. Prove that the relation thus introduced is an equivalence relation.

Being an equivalence relation, the above relation on the set of all bases decomposes it into two classes of equivalent bases.

Example. How do we decide, without calculating the matrix, whether two bases of the line (of the plane, or of the 3-space) are equivalent? We assume that the bases are given by a picture. The answer is simple. Two bases of the line (*i.e.* two vectors) are equivalent if they are co-directed. To each basis of the plane we can assign the rotation from the first vector to the second one along the smaller angle. Two bases are equivalent if either both of them are positive (*i.e.* are counterclockwise) or both are negative (clockwise). Finally, all bases of the 3-space can be decomposed into the left ones and the right ones, depending on whether the rotation from the first vector to the second one in the direction of the smaller angle is positive or negative when looked upon from the end of the third vector. Two bases are equivalent if they are of the same type. To determine the type of a basis, one could use the physicists' "screwdriver rule".

Exercise 4. Figure 1 shows three right bases and a left one. Find the left one.

Definition. *Orientation* of the space $\mathbb{R}^N$ is a class of equivalent bases.

Orientation is usually given by specifying a basis representing the relevant equivalence class.

Exercise 5. Prove that $\mathbb{R}^N$ has precisely two distinct orientations.

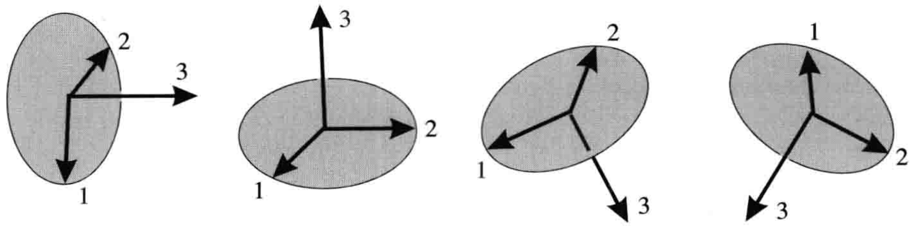


Figure 1. Right bases and a left one in $\mathbb{R}^N$.

It is also convenient to stipulate that the space $\mathbb{R}^0$ (the point) has two orientations, the orientation "+" and the orientation "–".

Definition. A system $a_0, a_1, \dots, a_n$ of $n + 1$ points in $\mathbb{R}^N$ is called *independent* if these points are not contained in the same plane of dimension $n - 1$ (or less).

We would like to stress that any system of n points is contained in some plane of dimension $\leqslant n - 1$.

Exercise 6. Prove that the independence of points $a_0, a_1, \dots, a_n$ is equivalent to the linear independence of the vectors $\overline{a_0a_1}, \overline{a_0a_2}, \dots, \overline{a_0a_n}$.

Exercise 7. Prove that any subset of an independent system of points is also an independent system of points.

Definition. The convex hull of $n+1$ independent points $a_0, a_1, \dots, a_n$ in $\mathbb{R}^N$ is called an *n-dimensional simplex*. The points $a_0, a_1, \dots, a_n$ are called the *vertices* of the simplex.

It follows from the definition that simplices of dimension 0, 1, 2, and 3 are points, segments, triangles, and tetrahedra, respectively.

The plane of the smallest dimension containing a given simplex is called the *support plane* of that simplex. Its dimension coincides with that of the simplex. *Orientation* of a simplex is an orientation of its support plane. It is given by the choice of a basis. According to our agreement on the orientations of $\mathbb{R}^N$, a 0-dimensional simplex, *i.e.* a point, has two possible orientations, "+" and "$-$".

Definition. A *face* of a simplex is the convex hull of some subset of the set of its vertices.

Exercise 8. Prove that a face of a simplex is itself a simplex.

Exercise 9. How many m-dimensional faces does an n-dimensional simplex have?

Exercise 10. What is the total number of faces of an n-dimensional simplex?

Definition. The *induced orientation* of an $(n-1)$-dimensional face of an oriented n-dimensional simplex is defined in the following way: we choose a basis of the n-dimensional simplex representing its orientation in such a way that the first $n-1$ vectors are contained in the given face and the remaining one is directed inside the simplex. Then the first $n-1$ vectors determine an orientation of the face. This rule is called "the rule of inward normal". See Figure 2 on the left. The induced orientations of the vertices of a one-dimensional simplex (a segment) are chosen such that the vector that orients the segment is directed from the plus to the minus.

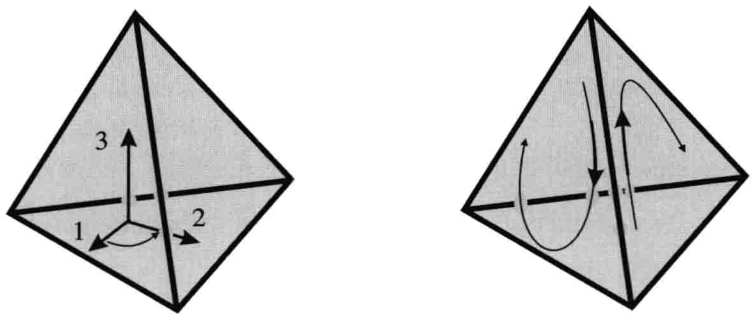


Figure 2. The induced orientation is defined by the rule of "inward normal". The iteration of this rule yields opposite orientations.

If α is the orientation of a simplex σ and δ is a face of it, then the induced orientation of δ is denoted as $\alpha|\delta$.

Theorem 2 (On doubly-induced orientations). *Let an $(n-2)$-dimensional simplex γ be a common face of $(n-1)$-dimensional faces δ_1, δ_2 of an n-dimensional simplex σ with orientation α. Then the orientations $(\alpha|\delta_1)|\gamma$ and $(\alpha|\delta_2)|\gamma$ are opposite.*

The proof of this theorem is obtained by a direct application of the definition of the induced orientation. Therefore we omit it, restricting ourselves to the illustration in Figure 2, right.

Definition. A finite collection of simplices in $\mathbb{R}^N$ is called a *simplicial complex* if any two of its simplices either have no common points or intersect along their common face.

We can stipulate that simplices without common points intersect along their common empty face. Then the above definition can be reduced to requiring that any two simplices intersect along their common face. We emphasize that, from the formal point of view, it is necessary to distinguish the notion of a simplicial complex (a collection of simplices) and of its *underlying space* (the union of these simplices). The underlying topological space of K is denoted by $|K|$. It is always a *polyhedron*, *i.e.* it can be presented as the union of some convex polytopes in $\mathbb{R}^N$. In this situation we say that the complex K *triangulates* the polyhedron $|K|$ (or represents a *triangulation* of it). The *dimension* of K is defined as the maximal dimension of its simplices.

Exercise 11. Give examples of simplicial complexes in the plane and in 3-space, as well as an example of a collection of simplices that does not form a simplicial complex.

Definition. An *orientation* of a simplicial complex is a set of orientations of each of its simplices including their faces.

Exercise 12. How many distinct orientations does the triangle, viewed as a simplicial complex, have?

The construction of the homology groups of a simplicial complex is carried out in two steps: first, to each simplicial complex we assign a certain so-called chain complex, then to this chain complex we assign its homology groups. From the methodological point of view it is more convenient to start with the second step.

1.3 Chain complexes

Definition. A sequence C of Abelian groups and their homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots,$$

infinite in both directions, is called a *chain complex* if for all n we have the equality $\partial_n \partial_{n+1} = 0$.

Let us stress that the equality $\partial_n \partial_{n+1} = 0$ should be understood in the following way: for any element x of the group C_{n+1} the element $\partial_n(\partial_{n+1}(x))$ should be the trivial element of C_{n-1}. We denote it by zero, since we employ the additive notation for the chain groups.

Definition. The group C_n is called the *n-dimensional chain group* of the complex C. The kernel $\operatorname{Ker} \partial_n \subset C_n$ of the homomorphism ∂_n is called the *group of n-dimensional cycles* and is denoted by A_n. The image $\operatorname{Im} \partial_{n+1} \subset C_n$ of ∂_{n+1} is called the *group of n-dimensional boundaries* of C and is denoted by B_n.

Exercise 13. Give an example of a sequence of groups and their homomorphisms that is not a chain complex.

Exercise 14. Find the groups of cycles and the groups of boundaries for all dimensions of the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots,$$

where the chain groups are given by the equalities $C_1 = \mathbb{Z}$, $C_2 = \mathbb{Z} \oplus \mathbb{Z}$, $C_n = 0$ for $n \neq 1, 2$, and the homomorphism ∂_2 is defined by the rule $\partial_2(m, n) = 3m + 3n$.

It is easy to show that for any chain complex the group of boundaries B_n is contained in the group of cycles A_n. The inverse is also true; if $B_n \subset A_n$ for all n, then the given sequence of groups and their homomorphisms is a chain complex, *i.e.* $\partial_n \partial_{n+1}$ is always 0.

Definition. The quotient group A_n / B_n is called the *n-dimensional homology group* of the chain complex C and is denoted by $H_n(C)$.

Exercise 15. Calculate the homology groups of the complex of Exercise 14.

Terminology. The elements of the group A_n are called *cycles* and those of B_n are called *boundaries*. The homomorphisms ∂_n are called *boundary homomorphisms*. Two cycles $a_1, a_2 \in A_n$ are called *homologous* if their difference $a_1 - a_2$ is a boundary, *i.e.* is an element of B_n. Thus, two cycles determine the same element of the homology group if and only if they are homologous. The elements of each homology group can be interpreted as classes of homology equivalent cycles.

Exercise 16. Calculate the homology groups of the elementary complex $E(m)$ which has the form

$$\cdots \longrightarrow 0 \xrightarrow{\partial_{m+1}} \mathbb{Z} \xrightarrow{\partial_m} 0 \longrightarrow \cdots$$

and whose chain groups are the following:

$$E_n(m) = \begin{cases} 0, & n \neq m, \\ \mathbb{Z}, & n = m. \end{cases}$$

Exercise 17. Calculate the homology groups of the elementary complex $D(m, k)$ which has the form

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_{m+1}} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

and whose chain groups are given by the rule

$$D_n(m, k) = \begin{cases} 0, & n \neq m, m+1, \\ \mathbb{Z}, & n = m, m+1, \end{cases}$$

and the homomorphism ∂_{m+1} consists in multiplication by an integer $k \neq 0$.

Exercise 18. Give a definition of the direct sum of chain complexes and prove that $H_n(C \oplus C') = H_n(C) \oplus H_n(C')$.

Definition. Let C and C' be two chain complexes. A family of homomorphisms $\varphi = \{\varphi_n \colon C_n \to C'_n, -\infty < n < \infty\}$ is called a *chain map* if $\varphi_n \partial_{n+1} = \partial_{n+1} \varphi_{n+1}$ for all n.

The meaning of the condition $\varphi_n \partial_{n+1} = \partial_{n+1}\varphi_{n+1}$ is that all the squares in the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & \\
\cdots \longrightarrow & C'_{n+1} & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow \cdots
\end{array}$$

are commutative.

Exercise 19. Let $\varphi\colon C \to C'$ be a chain map. Prove that $\varphi_n(A_n) \subset A'_n$ and $\varphi_n(B_n) \subset B'_n$, *i.e.* that φ takes cycles to cycles and boundaries to boundaries.

Theorem 3. *Let $\varphi\colon C \to C'$ be a chain map between chain complexes. Then for any integer n assigning to each cycle $x \in C_n$ the chain $\varphi_n(x) \in C'_n$ induces a well-defined homomorphism $\varphi_*\colon H_n(C) \to H_n(C')$.*

Proof. This theorem is almost obvious. Its proof does not present any difficulties, especially if the reader has completed Exercise 19. Nevertheless let us describe explicitly how a chain map φ between two chain complexes induces homomorphisms φ_* between the homology groups of matching dimensions. Here we encounter for the first time the so-called *diagrammatic search* (which is our preferred way to refer to the approach that is also known as "general nonsense"). This method is just a collection of some more or less standard tricks applied to diagrams. It is best to observe the method in practice. Let us reproduce the above diagram having removed, for simplicity, all the indices, see Figure 3. The elements used in the process of the proof are placed next to the groups to which they belong.

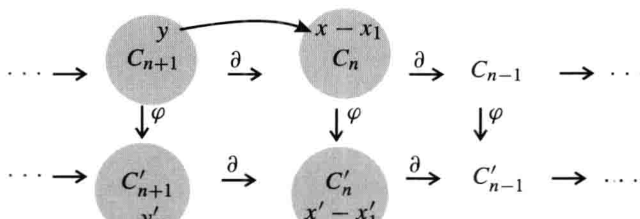


Figure 3. A proof via the method of diagrammatic search.

Let h be an arbitrary element of $H_n(C)$. We would like to use the given chain map φ to assign to h a well-defined element $h' = \varphi_*(h)$ of $H_n(C')$. The first step consists in choosing some cycle $x \in A_n \subset C_n$ representing h. Applying to x the homomorphism φ, we obtain a certain element $x' = \varphi(x)$ of the group C'_n.

Let us show that x' is a cycle. Indeed, $\partial x' = \partial\varphi(x) = \varphi(\partial x) = \varphi(0) = 0$ (we have used the commutativity of the diagrams and the fact that x itself is a cycle). Now we can define h' as the class containing the cycle x'.

Let us prove that the element h' of the group $H_n(C')$ does not depend on x, *i.e.* on the choice of an element representing h. Let x_1 be another representative, and let $x_1' = \varphi(x_1)$ be its image in C_n'. Denote by h'' the equivalence class containing x_1'. Then the difference $x - x_1$ is a boundary. Therefore there exists an element y of C_{n+1} such that $\partial y = x - x_1$. Again using the commutativity, we obtain that $x' - x_1' = \varphi(x - x_1) = \varphi\partial(y) = \partial\varphi(y) = \partial y'$, where $y' = \varphi(y)$. It follows that x' and x_1' differ by a boundary element, which means that the elements h' and h'' coincide. □

Exercise 20. Describe the category of all chain complexes and the category of sequences of Abelian groups. Check that assigning to each chain complex the sequence of its homology groups, and assigning to each chain map φ between chain complexes the induced map φ_* between their homology groups, yield together a functor from the former of the above two categories to the latter.

1.4 Homology groups of a simplicial complex

Let K be an oriented simplicial complex. We assign to it a chain complex $C(K)$ as follows. The elements of the n-dimensional chain group $C_n(K)$ are formal linear combinations of the form $m_1\sigma_1 + m_2\sigma_2 + \cdots + m_k\sigma_k$, where m_i are integers and $\sigma_1, \ldots, \sigma_k$ are all the n-dimensional simplices. The addition is coordinate-wise. Of course, the set $C_n(K)$ is a group with respect to this operation.

From the algebraic point of view, $C_n(K)$ is the free Abelian group that is freely generated, in an obvious sense, by the set of all the n-dimensional simplices. In particular, its rank is equal to the number of these simplices. Furthermore, suppose that there are no n-dimensional simplices in K. This may happen if n is negative or greater than the dimension of K. Then there are no linear combinations either. In this case we set $C_n(K) = 0$.

To define homomorphisms $\partial_n\colon C_n(K) \to C_{n-1}(K)$ it is sufficient to define the images of the generators, *i.e.* of all the simplices. Let σ be an n-dimensional simplex of K. Then each of its $(n-1)$-dimensional faces has two orientations, its own orientation that is a part of the total orientation of K and the orientation induced on it as a face of σ. We set by definition

$$\partial_n(\sigma) = \sum_{\delta_i \in K} \varepsilon_i \delta_i,$$

where the summation is over all the simplices δ_i of dimension $n-1$ and the numbers ε_i (called the *incidence coefficients*) are given by the following rule:

$$\varepsilon_i = \begin{cases} 0, & \text{if } \delta_i \text{ is not a face of } \sigma; \\ 1, & \text{if } \delta_i \text{ is a face of } \sigma \text{ and the two orientations coincide;} \\ -1, & \text{if } \delta_i \text{ is a face of } \sigma \text{ and the two orientations are distinct.} \end{cases}$$

The geometric meaning of this rule is quite simple. Recall that σ denotes not just an oriented simplex but also the chain $1 \cdot \sigma$ (an element of $C_n(K)$). Let us postulate that the chain $-\sigma = (-1) \cdot \sigma$ corresponds to the same simplex σ, but taken with the opposite orientation. Then $\partial_n(\sigma)$ is nothing more than the boundary of σ, where all the $(n-1)$-dimensional simplices contained in it are taken with their induced orientations.

Theorem 4. *For any simplicial complex K the groups $C_n(K)$ and the homomorphisms $\partial_n : C_n(K) \to C_{n-1}(K)$ form a chain complex (which we denote by $C(K)$).*

The proof of this theorem follows from Theorem 2 on doubly-induced orientations, which we now can rephrase as follows: the boundary of a boundary is empty. The fundamental importance of this fact, which lies in the foundation of any homology theory, merits a careful and deep consideration.

Definition. Let K be an oriented simplicial complex. Then the homology groups of the corresponding chain complex $C(K)$ are called the *homology groups of K* and are denoted by $H_n(K)$.

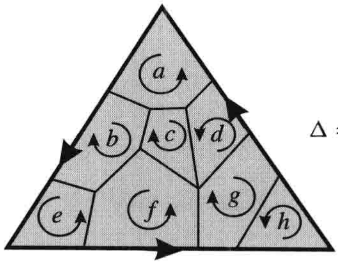
In other words, the group $H_n(K)$ is the quotient group $\operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ of the kernel of the homomorphism ∂_n by the image of the homomorphism ∂_{n+1}.

Exercise 21. Prove that the groups $H_n(K)$ do not depend on the choice of an orientation of K.

One can also prove that the homology groups of any *polyhedron* (a subset of $\mathbb{R}^N$ which can be presented as a simplicial complex) do not depend on any particular choice of such presentation, *i.e.* on the triangulation. The proof of this result is rather cumbersome, although it does not present serious difficulties of theoretical nature. For instance, one may proceed as follows:

1. Make sure that the above construction of the homology groups can be carried over to polyhedra decomposed not necessarily into simplices but into arbitrary polytopes.

2. Prove that if some decomposition of a triangulated polyhedron K into polytopes has the property that each simplex consists of whole polytopes, then the homology groups calculated via the triangulation are isomorphic to those calculated via the decomposition into polytopes. It is easiest to describe the desired isomorphism using the generators of the chain groups of the triangulation, *i.e.* simplices. To each n-dimensional simplex σ of the triangulation we assign the chain that consists of all the polytopes comprising σ. The coefficients at those polytopes are equal to ± 1, depending on whether the orientation of a given polytope coincides with that of σ or is opposite to it. Therefore the boundary of this chain coincides with the boundary of the simplex, which essentially ensures the isomorphism of the homology groups, see also Figure 4.



$$\Delta = a - b - c + d + e + f - g + h$$

Figure 4. An oriented simplex is assigned the chain composed of the polytopes comprising it.

3. For any two given triangulations of the same polyhedron we may consider the decomposition into the polytopes given by intersections of any two simplices belonging to the two triangulations. Then the previous item ensures the desired isomorphism of the homology groups.

It is important to mention that it is possible to define the homology groups not only for polyhedra but also for more general spaces homeomorphic to polyhedra. Sometimes such spaces are called *topological polyhedra*. Topological polyhedra can by definition be triangulated, but into *curvilinear simplices* (images of genuine simplices under the relevant homeomorphism).

Thus, in order to calculate the homology groups of a given topological space, one should perform the following steps:

1. Present the space as a polyhedron and triangulate it.

2. Choose an orientation for the simplicial complex thus obtained.

3. Calculate the chain groups C_n.

4. Describe the boundary homomorphisms ∂_n.

5. Calculate the groups of cycles A_n.

6. Calculate the groups of boundaries B_n.

7. Calculate the quotient groups $H_n = A_n/B_n$.

Theorem 5. *The homology groups of the point are the following:*

$$H_n(*) = \begin{cases} 0, & n \neq 0, \\ \mathbb{Z}, & n = 0. \end{cases}$$