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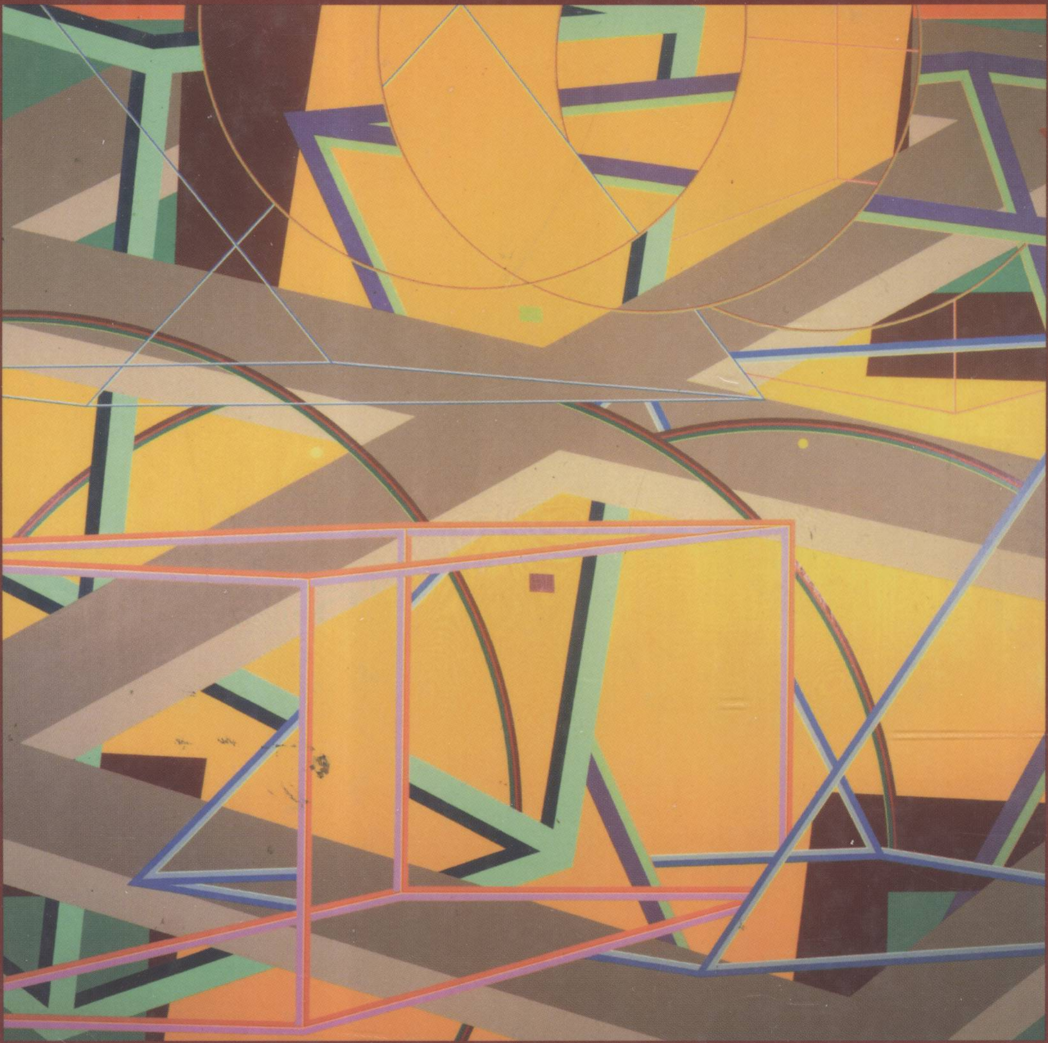
Geometry and Its Applications



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Geometry and Its Applications

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Cover image: Al Held, C-Y-I, 1978, acrylic on canvas, 114×114″, Albright-Knox Art Gallery, Buffalo, New York, Gift of Seymour H. Knox, 1979

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ACADEMIC PRESS
A division of Harcourt Brace & Company
525 B Street, Suite 1900, San Diego, CA 92101-4495, USA
http://www.apnet.com

Academic Press
24–28 Oval Road, London NW1 7DX, UK
http://www.hbuk.co.uk/ap/

Harcourt/Academic Press
200 Wheeler Road
Burlington, MA 01803
http://www.harcourt-ap.com

Library of Congress Number: 98-28224

International Standard Book Number: 0-12-493270-3

Printed in the United States of America
98 99 00 01 02 IP 9 8 7 6 5 4 3 2 1

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Preface

This book is for a first college-level geometry course and is suitable for mathematics majors and especially prospective high school teachers. Much of the content will be familiar to geometry instructors: a solid introduction to axiomatic Euclidean geometry, some non-Euclidean geometry, and a substantial amount of transformation geometry. However, we present some important novelties: We pay significant attention to applications, we provide optional dynamic geometry courseware for use with *The Geometer's Sketchpad*, and we include a chapter on polyhedra and planar maps. By extending the content of geometry courses to include applications and newer geometry, such a course can not only teach mathematical skills and understandings, but can help students understand the twenty-first century world that is unfolding around us. By providing software support for discovery learning, we allow experiments with new ways of teaching and learning.

The intertwined saga of geometric theory and applications is modern as well as ancient, providing a wonderful mathematical story that continues today. It is a compelling story to present to students to show that mathematics is a seamless fabric, stretching from antiquity until tomorrow and stretching from theory to practice. Consequently, one of our goals is to express the breadth of geometric applications, especially contemporary ones. Examples include symmetries of artistic patterns, physics, robotics, computer vision, computer graphics, stability of architectural structures, molecular biology, medicine, pattern recognition, and more. Perhaps surprisingly, many of these applications are based on familiar, long-standing geometric ideas — showing once again that there is no conflict between the timelessness and modernity of good mathematics.

In recent years, high school instruction in geometry has become much less extensive and much less rigorous in many school districts. Whatever advantage this may bring at the high school level, it changes the way we need to instruct mathematics students at the college level. In the first place, it makes instruction in geometry that much more imperative for all students of mathematics. In addition, we cannot always assume extensive familiarity with proof-oriented basic Euclidean geometry. Consequently, we begin at the beginning, displaying a portion of classical Euclidean geometry as a deductive system. For the most part, our proofs are in the style of Euclid — which is to say that they are not as rigorous as they could be. We do present a snapshot of some geometry done with full rigor, so that students will have exposure to that. In addition, there is a careful discussion of why full rigor is important in some circumstances and why it is not always attempted in teaching, research, or applications.

Except for Chapters 5 and 6 and parts of Chapter 7, this text requires little more than high school mathematics. Nonetheless, students need the maturity to deal with proofs and careful calculations. In Chapters 5 and 6 and parts of Chapter 7, we assume a familiarity with vectors as commonly presented in multivariable calculus. Derivatives also make a brief appearance in Chapter 5. Matrices are used in Chapter 6, but it is not necessary to have studied linear algebra in order to understand this material. All that we assume is that students know how to multiply matrices and are familiar with the associative law.

It would be foolish to pretend that this book surveys all of the major topics and applications of geometry. For example, differential geometry is represented only by one short section. I have tried to choose topics that would be most appealing and accessible to undergraduates, especially prospective high school teachers.

A good deal of flexibility is possible in selecting a sequence of topics from which to create a course. This book contains two approaches to geometry: the axiomatic and the computational. When I am teaching mostly prospective teachers, I emphasize the axiomatic (Chapters 1–4) and sprinkle in a little computational material from Chapters 5 or 6. When I have mainly mathematics majors with applied interests and others, such as computer science majors, I reverse the emphasis, concentrating on Chapters 5, 6, and 7. I find Chapter 8 works well in either type of course.

There is a lot of independence among the chapters of the book. For example, one might skip Chapters 1 through 3 since there are only a few places in other chapters (mainly Chapter 4) where there is any explicit dependence on them. An instructor can remind students of the relevant theorems as the need arises. Chapter 4 is not needed for any of the other parts. Chapter 5 can be useful in preparation for Chapter 6 only insofar as we often think of points as position vectors in Chapter 6. Chapter 7 relies on one section of Chapter 2 and one section of Chapter 5. Chapter 8 is completely independent of the other chapters. More detailed descriptions of prerequisites are given at the start of each chapter.

In writing this book, I am aware of the many people and organizations that have shaped my thoughts. I learned a good deal about applications of geometry at the Grumman Corporation (now Northrop-Grumman) while in charge of a robotics research program. Opportunities to teach this material at Adelphi University and during a year spent as a visiting professor at the U.S. Military Academy at West Point have been helpful. In particular, I thank my cadets and my students at Adelphi for finding errors and suggesting improvements in earlier drafts. Thanks are due to the National Science Foundation, the Sloan Foundation, and COMAP for involving me in programs dedicated to the improvement of geometry at both the collegiate and secondary levels. Finally, I wish to thank numerous individuals with whom I have been in contact (for many years in some cases) about geometry in general and this book in particular: Joseph Malkevitch, Donald Crowe, Robert Bumcrot, Andrew Gleason, Greg Lupton, John Oprea, Brigitte Selvatius, Marie Vanisko, and Sol Garfunkel.

Prof. Walter Meyer
Adelphi University

Supplements for the Instructor

The following supplements are available from Academic Press:

1. Answers to the even-numbered exercises
2. Instructor's guide to *The Geometer's Sketchpad* explorations that are contained in the disk that accompanies this text.

Introduction

Geometry is full of beautiful theorems, and its logical structure can be inspiring. As the poet Edna St. Vincent Millay wrote, "Euclid alone has looked on beauty bare." But beyond beauty and logic, geometry also contains important tools for applied mathematics. This should be no surprise, since the word "geometry" means "earth measurement" in Greek. As just one example, we will illustrate the appropriateness of this name by showing how geometry was used by the ancient Greeks to measure the circumference of the earth without actually going around it. But the story of geometric applications is modern as well as ancient. The upsurge in science and technology in the last few decades has brought with it an outpouring of new questions for geometers. In this introduction we provide a sampler of the big ideas and important applications that will be discussed in this book.[1]

Individuals often have preferences, either for applications in contrast to theory or vice versa. This is unavoidable and understandable. But the premise of this book is that, whatever our preferences may be, it is good to be aware of how the two faces of geometry enrich each other. Applications can't proceed without an underlying theory. And theoretical ideas, although they can stand alone, often surprise us with unexpected

[1] This introduction also appears in *Perspectives on the Teaching of Geometry for the 21st Century*, ed. V. Villani and C. Mammana, copyright Kluwer Academic Publishers b.v., 1998.

applications. Throughout the history of mathematics, theory and applications have carried out an intricate dance, sometimes dancing far apart, sometimes close. My hope is that this book gives a balanced picture of the dance at this time, as we enter a new millennium.

Axiomatic Geometry

Of all the marvelous abilities we human beings possess, nothing is more impressive than our visual systems. We have no trouble telling circles apart from squares, estimating sizes, noticing when triangles appear congruent, and so on. Despite this, the earliest big idea in geometry was to achieve truth by proof and not by eye. Was that really necessary or useful? These ideas are explored in Chapters 1 and 2.

Creating a geometry based on proof required some basic truths — which are called *axioms* in geometry. Axioms are supposed to be uncontroversial and obviously true, but Euclid seemed nervous about his parallel axiom. Other geometers caught this whiff of uncertainty and, about 2000 years later, some were bold enough to deny the parallel axiom. In doing this they denied the evidence of their own eyes and the weight of 2000 years of tradition. In addition, they created a challenge for students of this so-called "non-Euclidean geometry," which asks them to accept axioms and theorems that seem to contradict our everyday visual experience. According to our visual experience, these non-Euclidean geometers are cranks and crackpots. But eventually they were promoted to visionaries when physicists discovered that the far-away behavior of light rays (physical examples of straight lines) is different from the close-to-home behavior our eyes observe. Astronomers are working to make use of this non-Euclidean behavior of light rays to search for "dark matter" and to foretell the fate of the universe. These revolutionary ideas are explored in Chapter 3.

Rigidity and Architecture

If you are reading this indoors, the building you are in undoubtedly has a skeleton of either wooden or steel beams, and your safety depends partly on the rigidity of this skeleton (see Figure I.1b). Neither a single rectangle (Figure I.1a), nor a grid of them, would be rigid if it had hinges where the beams meet. Therefore, when we build frameworks for buildings, we certainly don't put hinges at the corners — in fact, we make these corners as strong as we can. But it is hard to make a corner perfectly rigid, so every additional safeguard is welcome. A very common safeguard, which makes a single rectangle rigid, is to add a diagonal brace. Perhaps surprisingly, if we have a grid of many rectangles, it is not necessary to brace every rectangle. The braced grid in

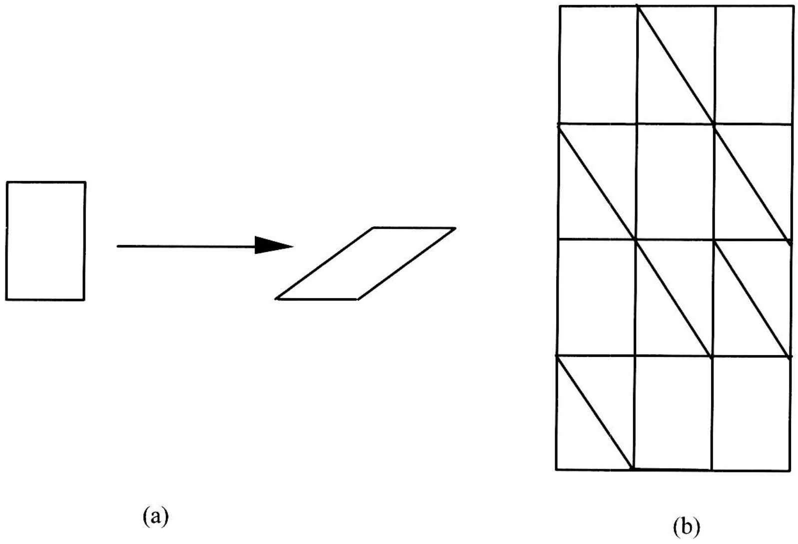


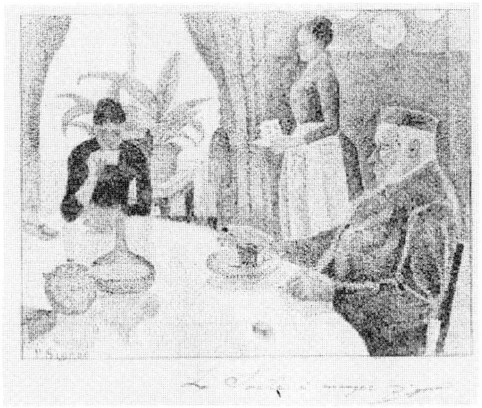
Figure I.1 (a) A hinged rectangle flexing. (b) A braced grid that cannot flex even if totally hinged.

Figure I.1b turns out to be rigid even if every corner is hinged. In Section 2.3 of Chapter 2, we work out a procedure for determining when a set of braces makes a grid of rectangles rigid even though all corners are hinged.

Computer Graphics

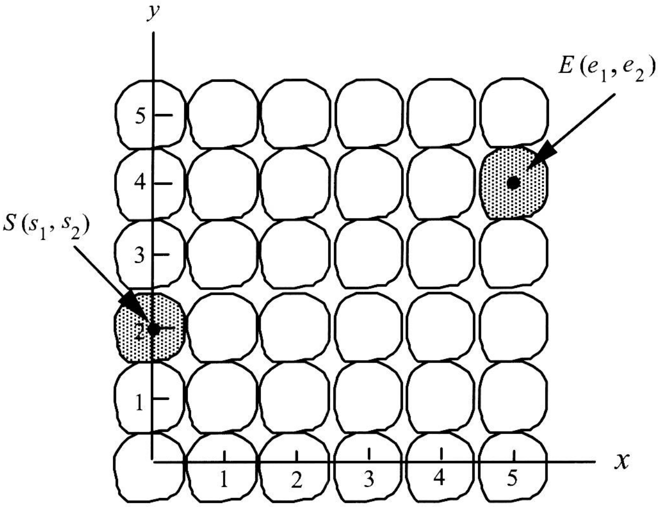
The impressionist painter Paul Signac (1863–1935) painted "The Dining Room" by putting lots of tiny dots on a canvas (Figure I.2a). If you stand back the tiny dots blend together to make a picture. This painting technique, called *pointillism*, was a sensation at the time, and foreshadows modern image technology. For example, if you take a close look at your TV screen, you'll see that the picture is composed of tiny dots of light. Likewise, a computer screen creates a picture by "turning on" little patches of color called *pixels*. Think of them as forming an array of very tiny light bulbs, arranged in rows and columns in the x-y plane so that each point with integer coordinates is the center of a pixel (Figure I.2b).

When a graphics program shows a picture, how does it calculate which pixels to turn on and what colors they should be? Here is a simple version of the problem: If we are given two pixels (shaded in Figure I.2b) and want to connect them with a set of pixels to give the impression of a blue straight line, which "in between" pixels should be turned



The Dining Room
by Paul Signac

(a)



(b)

Figure I.2 Pixels in (a) art and (b) computer graphics. Courtesy of Metropolitan Museum of Art.

blue? Neither Signac nor you would have any problem with this, painting by eye, but how can the computer do it by calculations based on the coordinates of the centers of the start and end pixels? In this computer verison of the problem, the desired answer is a list of pixel centers, each center specified by x and y coordinates. Chapter 1 gives you an idea of how to create such a list.

Symmetries in Anthropology

In studying vanished cultures, anthropologists often learn a great deal from the artistic patterns these cultures produced. Figure I.3 shows two patterns you might find on a cloth or circling around a clay pot. Each of these patterns has some kind of symmetry. But what kind? What do we mean by the word *symmetry*? Some would say the bottom pattern has more symmetry than the top pattern. How can symmetry be measured? The questions we pose here for patterns are similar to ones that arise, in three-dimensional form, in the study of crystallography.

We study these questions in Chapter 4, with particular attention paid to the pottery of the San Ildefonso pueblo in the southwestern United States.

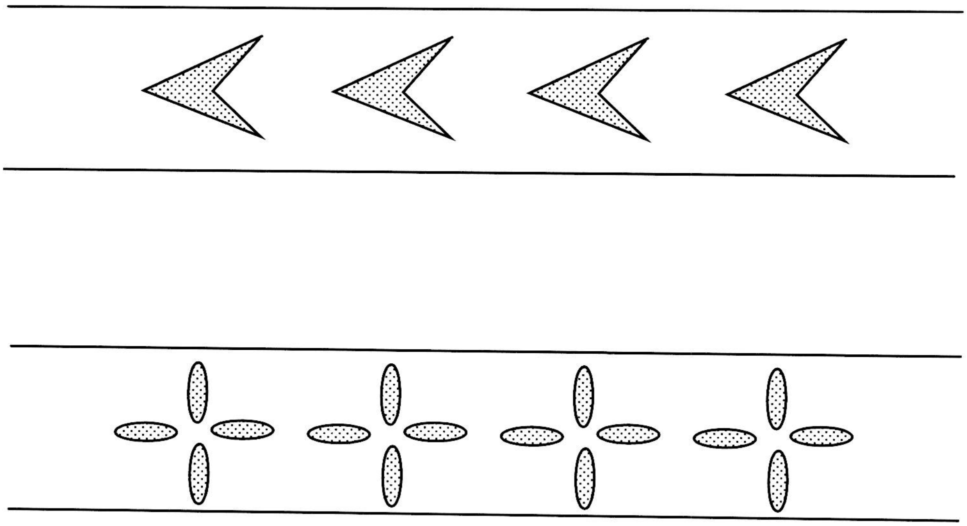


Figure I.3 Two strip patterns.

Robotics

Figure I.4 shows a robot about to drill a hole. Perhaps the hole is being drilled into someone's skull in preparation for brain surgery, or maybe it's part of the manufacture of an automobile. Whatever the purpose, the drill tip has to be in just the right place and pointing just the right way. The robot moves the drill about by changing its joint angles. If we specify the x, y, z coordinates of the drill tip, how do we calculate the values of θ_1, θ_2, and θ_3 needed to bring the drill tip to the desired point? Can we also specify the direction of the drill? These are questions in *robot kinematics*. In Section 6.5 of Chapter 6, we study the basics of robot kinematics for a two-dimensional robot. This is the same kind of mathematics used for three-dimensional robot kinematics.

Molecular Shapes

As chemistry advances, it pays more attention to the geometric shapes of molecules. In 1985 a new molecular shape was discovered, called the *buckyball*[2], that reminded chemists of the pattern on a soccer ball. The molecule consisted of 60 carbon atoms distributed in

[2] Named for the architect Buckminster Fuller, who promoted the idea of *geodesic domes*, buildings in the shape of unusual polyhedra.

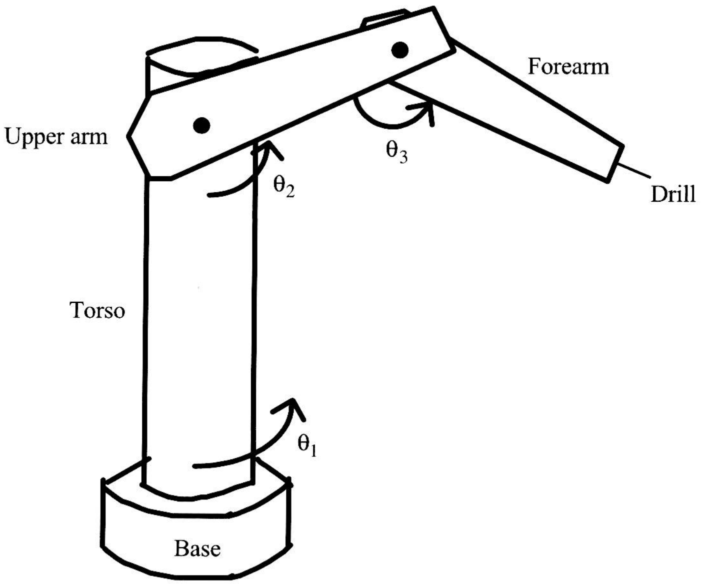


Figure I.4 A simplified version of a PUMA robot arm.

a roughly spherical shape—like the corners of the pattern on the soccer ball. Coincidentally, this pattern had been discovered not only by soccer ball manufacturers, but many centuries ago by mathematicians. Figure I.5 shows the pattern in a Renaissance drawing of a truncated icosahedron, by Leonardo da Vinci. To understand the structure of the buckyball, think of the corners of the truncated icosahedron as being occupied by carbon atoms and think of the connecting links as representing chemical bonds betweeen certain carbon atoms. Each carbon atom is connected to each of three other nearby carbon atoms with a chemical bond. This pattern contains 12 pentagons and 20 hexagons.

Once chemists discovered this molecule, they looked for other molecules involving just carbons, where each carbon is connected to exactly three others and where the pattern has only hexagons and pentagons. Each such molecule turns out to have 12 pentagons. This is not a chemical quirk! It follows from some mathematics we study in Section 8.4 of Chapter 8.

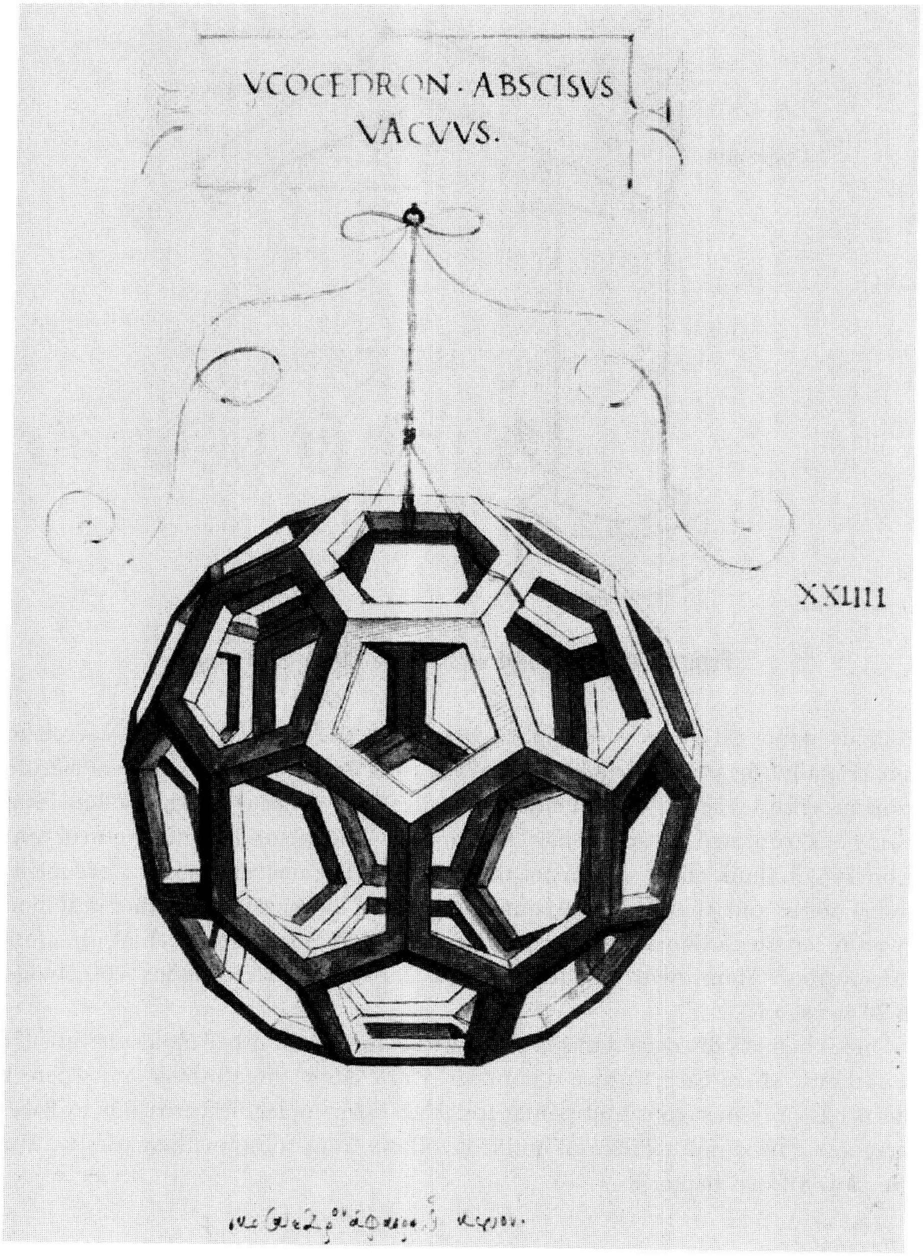


Figure I.5 The pattern of the buckyball molecule, drawn five centuries before the discovery of the molecule! Courtesy of Jerry Blow.

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