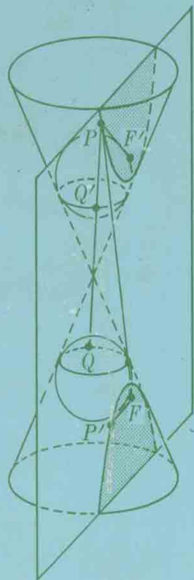
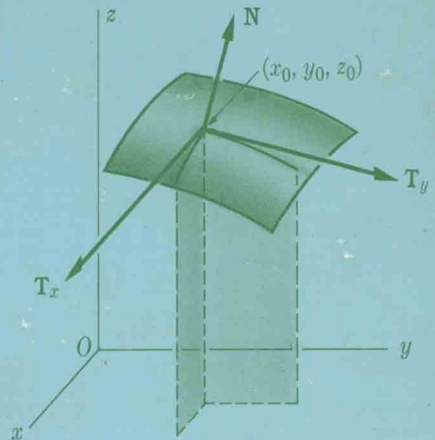




ANALYTIC GEOMETRY: A VECTOR APPROACH

CHARLES WEXLER

ANALYTIC GEOMETRY
A Vector Approach

by

CHARLES WEXLER
Department of Mathematics
Arizona State University

ADDISON-WESLEY PUBLISHING COMPANY
Reading, Massachusetts · Menlo Park, California · London · Don Mills, Ontario

Printed in the United States of America

Library of Congress Catalog Card No. 61-10974

Third Printing—July 1967

PREFACE

In my own teaching I have found that to introduce the student to analytic geometry by way of vectors not only furnishes elegant proofs but also gives him early familiarity and training in vector concepts that are invaluable to future scientific work. By means of vectors, solid analytic geometry can be studied practically simultaneously with plane analytic geometry, and polar coordinates can be brought in early. Not the least of many advantages is that early introduction to vector methods in analytic geometry makes later work in physics or engineering much easier, and in fact would save much time in those courses that must otherwise be devoted to developing vectors.

A recent trend in mathematics has been to combine, or "integrate," analytic geometry and calculus, usually in a three-semester sequence. Most texts now available which combine these two subjects underemphasize the analytic geometry. After a very meager introduction to analytic geometry, they move into calculus at full speed, not returning to analytic geometry until 150 or 200 pages farther on. It seems to me that this scanty preparation in analytic geometry is not an adequate background for the calculus. Perhaps I may be considered too conservative, but I have felt that the emphasis during the first semester, or at least the first quarter, should be on the analytic geometry, with calculus first being developed for finding tangents to curves and normals to surfaces; and that there should be too much analytic geometry to choose from rather than too little. This book was originally begun with this aim in mind: to combine analytic geometry and calculus in this manner for a three-semester course, but when the analytic geometry subject matter was completed together with the necessary calculus for tangents and normals, it seemed appropriate to publish it as a straight analytic geometry text for a course of one semester. Incidentally, only the difference quotient limit is used; calculus language such as *derivative*, D_xy, etc. is avoided. The constant repetition of its use emphasizes that the difference quotient is a basic concept of the calculus.

These three features, the thorough development of vector methods from the very beginning, the generous amount of analytic geometry to choose from, and the not inconsiderable amount of calculus motivated by the need to find tangents and normals, may so enhance the value of the regular analytic geometry course as to save it from fading into oblivion.

This book is not, and is not intended to be, an exhaustive treatise on vectors. It uses only that much of vector analysis that goes naturally with the development of analytic geometry and calculus at the elementary level. A remarkably small amount of vector analysis is needed for this purpose, only up to and including the dot and cross products of vectors and their geometric properties.

It was thought worthwhile to include a little linear algebra, so some material on abstract vector spaces is woven in, paralleling the three-dimensional development to a certain extent, up to the point of defining inner product spaces. This material may be too difficult for most beginning students, but some instructors may wish to experiment to see how much of it can be absorbed. It is purely background material, and no exercises are based on it. I shall appreciate receiving comments as to how successful this feature is.

Incidentally, some knowledge of determinants is needed, and a short summary of their properties and main theorems is given, mostly without proof. In several places in the text, the conditions for nontrivial solutions of linear homogeneous equations are needed, and these are developed more fully (with proofs), together with Cramer's Rule.

Every instructor knows how "unfair" of him it is to expect students to remember any trigonometry after they have passed that course, so the student will have to be prodded into recalling certain exotic items such as the functions of 0°, 30°, 45°, 60°, 90°, 180°, the addition formulas, the double-angle formulas, right-triangle relations, etc., as they are needed. Some of these topics are reviewed here and there in the text, some from the point of view of vectors.

It is very important that students be given regular out-of-class assignments and that some method be found to insure that these are seriously attempted, because then the class explanations take on much more meaning for the student. It is especially important in the case of the vector assignments that all or nearly all of the problems not only be assigned, but also be done in class by the instructor. Many of the problems extend the theory. Vectors are a strange and new subject for the student; it is not a forbidding subject, in fact most students are quite intrigued and often fascinated by the power and elegance of vectors; but since vectors are new and unfamiliar, the student should be led by the hand a bit more than usual. After all, the text states a fact only once or twice, while the repetition of the basic ideas given by the instructor in doing the problems in class helps enormously in clarifiying vectors for the student and makes them appear natural to use.

There is probably more material in the book than can be handled comfortably in one semester. The general equation of the second degree and abstract vector spaces could be postponed to a later course (or assigned as a reading project for the good students), and the material on radical axes could be omitted. Discussions of families of curves could be omitted. Also, if time is limited, fewer exercises could be done in class, especially those on the fine points concerning conic sections. But the vector material at the beginning and the calculus material at the end should not be curtailed.

I wish to take this opportunity to express my deep appreciation and gratitude to Mr. M. L. Clabaugh, who spent many hours typing and retyping the manuscript and who suggested many improvements. My thanks also to several anonymous reviewers for their constructive criticism and suggestions, most of which I tried to follow.

C. W.

CONTENTS

CHAPTER 1

INTRODUCTION

Analytic geometry is a wedding of algebra and geometry. The ancient Greek mathematicians, with their marvelous minds, concentrated on geometry and perfected it to a high degree. But, to our great loss, they more or less ignored arithmetic (and the beginnings of algebra) because it was developed to serve trade and commerce. Trade and accounts, and hence arithmetic, were the province of slave supervisors, while mathematicians devoted their thoughts to the higher arts. It took about 1500 years for arithmetic and algebra to spread across Europe, and it remained for René Descartes to recognize what a powerful tool algebra would be in simplifying and extending geometry. It was only after a sufficient interval for developing this idea that the time was ripe for Isaac Newton to create an even more powerful new tool, the calculus.

Descartes' fundamental idea, like most great inventions, was simple. Why not set up two basic directions, such as a "horizontal" and a "vertical" direction (from a central starting point O, called the *origin*), which would serve as measuring sticks to locate any desired point P in the plane? The distance along the horizontal direction could be denoted by a negative, positive, or zero number, depending on whether one had to go to the left, or to the right, or neither to reach a given point. Similarly, the number denoting the vertical distance could be positive, negative, or zero, according to whether one had to go up, down, or neither to reach the point.

Following Descartes' construction, illustrated in Fig. 1–1, we find that to every point in the plane there corresponds a pair of real numbers, x and y; and, conversely, to every pair of real numbers there corresponds one

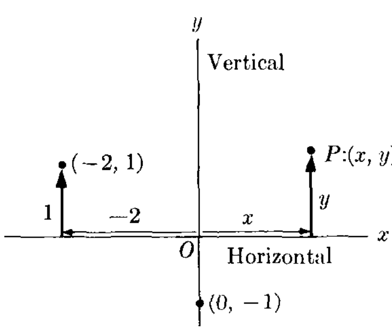


FIGURE 1–1

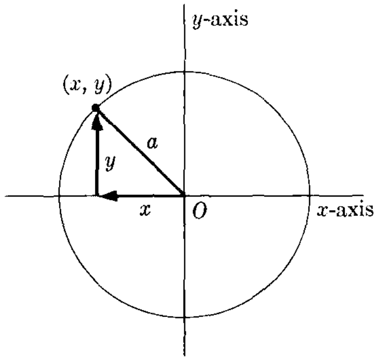


FIGURE 1–2

and only one point. The horizontal line is called the x-axis, its positive direction being to the right; the vertical line is called the y-axis, its positive direction being up. The two numbers written "(x, y)" are called the *coordinates* of the point P, respectively the x-coordinate and y-coordinate (sometimes called the abscissa and the ordinate). The origin O has coordinates $(0, 0)$; any point on the x-axis has coordinates $(a, 0)$; just as any point on the y-axis has coordinates $(0, b)$. Thus the x-axis is characterized by the equation $y = 0$. In words, every point on the x-axis has as its y-coordinate the value of zero; conversely, any point whose y-coordinate is zero lies on the x-axis.

When there is such a relation between a curve and an equation, namely, that the coordinates of any point on the curve satisfy the equation and any point whose coordinates satisfy the equation must lie on the curve, we say that the equation is the *equation of the curve*. Thus $y = 0$ is the *equation* of the x-axis. Similarly, $x = 0$ is the *equation* of the y-axis.

It can be seen that the idea of a "coordinate system" or "frame of reference," as this scheme of having an origin and two basic directions is called, presents great possibilities. For example, to study a circle of radius a, we can choose the origin to be at the center of the circle (Fig. 1–2). We see by the Pythagorean theorem that the coordinates (x, y) of any point on the circle satisfy the equation $x^2 + y^2 = a^2$, and that any point whose coordinates satisfy the equation $x^2 + y^2 = a^2$ is the vertex of a right triangle of hypotenuse a, at distance a from the origin, and hence lies on the circle. All properties of a circle are presumably embodied in its equation. We shall study the circle in detail from this point of view later.

Going back to Descartes' fundamental idea, we see that we can reach any point in the plane by means of two *directed* line segments, right or left and up or down. Similarly, we can reach any point in three-dimensional space with three directed line segments by providing a third axis,

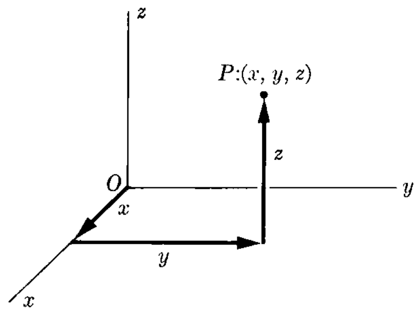


FIGURE 1–3

the z-axis, perpendicular to the plane of the x-axis and the y-axis at the origin (Fig. 1–3).

The basic idea of directed line segments is very important, not only in analytic geometry and mathematics in general, but also in physics and engineering. Directed line segments are called *vectors*. We shall devote the next chapter to building up an "algebra" of vectors and using it to develop the fundamental concepts of analytic geometry. The reader will find vector analysis a somewhat strange but very stimulating subject. If he applies himself and learns to incorporate it into his thinking, he will find it a wonderful help in finding his way about in space. But the thrill of knowing that he is beginning to master the subject will come only when he finds he can do many of the exercises, some of which are not easy.

EXERCISE GROUP 1–1

1. Draw a two-dimensional frame of reference, marking the x- and y-axes. Plot the following points and write the coordinates next to each point in your drawing: $(3, 0)$, $(0, 2)$, $(-3, -2)$, $(-2, 3)$.

2. Draw a three-dimensional frame of reference, marking the three axes. Plot the following points: $(2, 0, 0)$, $(0, 1, 0)$, $(1, 2, 3)$.

3. What is the equation of the straight line through $(2, 1)$ and $(2, -3)$? through $(2, 1)$ and $(-1, 1)$? Draw each line first.

4. What is the equation of the xy-plane in three dimensions? the xz-plane? the yz-plane? the plane through the points $(3, 1, 2)$, $(-1, 2, 2)$, $(0, 0, 2)$?

5. Find two equations that together characterize completely the x-axis in three dimensions; likewise for the y-axis; the z-axis.

6. Find the equation of the line containing the origin and the point $(1, 1)$.

7. $x^2 - 9 = 0$ is the equation of what curve(s)?

8. $x^2 + y^2 - 9 = 0$ is the equation of what curve?

A remark is in order about the use of our phrase "the equation of a curve." It was pointed out above that $y = 0$ is the equation of the x-axis. The y-coordinate of every point on the x-axis is certainly zero, and any point whose y-coordinate equals zero certainly lies on the x-axis. But the same is true of $5y = 0$ or of $(x^2 + 1)y = 0$. Every point on the x-axis satisfies each of these equations; also, any point that satisfies either of these equations must lie on the x-axis. The reason is that $5y$ or $(x^2 + 1)y$ can equal zero only when $y = 0$. So, strictly speaking, $y = 0$ is only one possible equation of the x-axis. There are many. Hence when we speak of "*the* equation" of a certain curve, we shall mean that the particular equation being discussed satisfies the two-part definition of "equation of a curve," namely, that every point on the curve satisfies the equation and, conversely, any point that satisfies the equation must lie on the curve.

CHAPTER 2

FUNDAMENTAL CONCEPTS

2–1 Vectors. Definitions and operations. A vector is a directed line segment. Note that there are two aspects to a vector: it has *direction,* and it has *length* or *magnitude.* Any entity that has these two qualities can be represented by a vector. Thus if we use a scale of $\frac{1}{4}$ inch = 10 miles, the speed of a car traveling northeast at 40 mi/hr can be expressed as a one-inch arrow pointing in a direction 45° above the horizontal to the right (east) (Fig. 2–1). The weight of a 165-lb man (the gravitational pull of the earth) can be represented by an arrow of appropriate length pointed in a downward direction.

In print, vectors are usually denoted by boldface type, such as **A** or **r** and their length by $|\mathbf{A}|$ or $|\mathbf{r}|$, sometimes called the *absolute value* of **A** or of **r**. In handwriting we usually indicate the vector by a letter with an arrow above it: $\vec{A}$, $\vec{r}$. Sometimes, even in print, we shall use a beginning point and an endpoint with an arrow, for example, $\overrightarrow{OP}$, to designate a vector; its magnitude is then designated by $|\overrightarrow{OP}|$. See Fig. 2–2.

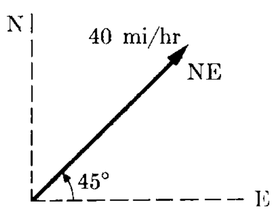


FIGURE 2–1

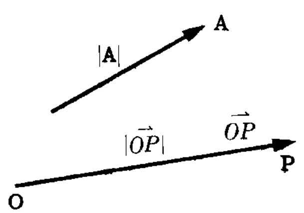


FIGURE 2–2

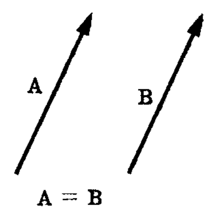
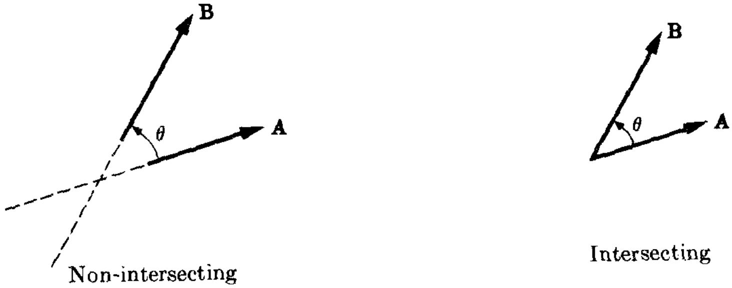


FIGURE 2–3

(a) *Definition of equality.* Two vectors are equal if and only if they have the same length and the same direction. Note (Fig. 2–3) that equal vectors do not necessarily start from the same point, nor must they be along the same line. The vectors we will deal with are *free* vectors, unless otherwise specified. They can be transported from place to place; and so long as they remain of the same length and keep the same direction, we consider them to be the same vector or equal vectors. This may seem to be a rather loose way of thinking of "same" or "equal." Some writers prefer the term "equivalent," reserving "equal" to mean identical or coincidental vectors. But we prefer to use the words "equal" or "same" in the loose sense; in the few cases where we mean to indicate strict equality, we shall be careful to point it out.



Non-intersecting

(a)

Intersecting

(b)

FIGURE 2–4

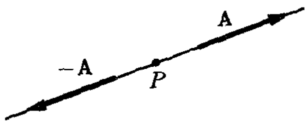


FIGURE 2–5

We shall use the word "parallel" with the same freedom. The term *parallel vectors* will include the special case where the two vectors are along the same line or even when they coincide. Thus a vector is parallel to itself. Also, two vectors in opposite directions, as well as vectors in the same direction, can be called parallel.

Consistent with the idea of free vectors is that of *angle* between two vectors, even if the vectors do not meet. We can transport them parallel to themselves to emanate from the same point. Then the angle between them that is between 0° and 180° inclusive is called the angle between the two vectors (Fig. 2–4b). Thus if one's right arm extends upward and the left arm forward, the angle between them is 90° in spite of the fact that they do not meet in a vertex. Since we cannot distinguish between clockwise and counterclockwise in three dimensions, we do not need negative angles.

One other term that we shall use occasionally is *direction vector.* Very often we are concerned only with the direction of a vector and not with its size. For example, a straight line is completely determined by a point P which lies on it and any vector $\mathbf{A}$ along it (Fig. 2–5). The magnitude of $\mathbf{A}$ does not matter; a vector twice as long, or in general $k\mathbf{A}$, would do just as well, where k is a pure number not zero and may even be negative. The pure number k is called a *scalar.* Likewise, a plane (Fig. 2–6) is determined by a given point P which lies in it and any vector $\mathbf{N}$ perpendicular to it. Such a vector is called a *normal* of the plane. $\mathbf{N}$ may be of any length; $k\mathbf{N}$ would do just as well. We shall freely multiply

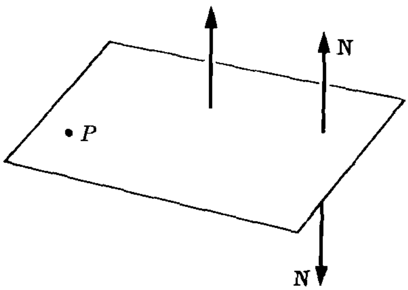


FIGURE 2–6

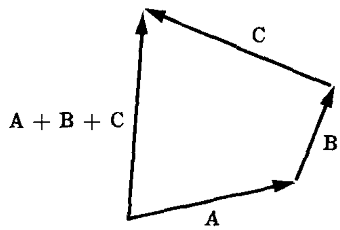


FIGURE 2–7

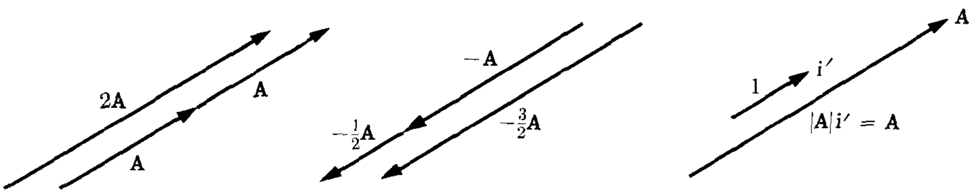


FIGURE 2–8

a direction vector by any convenient scalar not zero; and just as we spoke of *the* equation of a line even though it was one of many, we shall speak of *the* direction vector of a line and *the* normal to a plane. In the next paragraph we discuss further the concept of a scalar times a vector.

(b) *Definition of addition.* Two or more vectors are added by the *polygon rule:* from the arrowhead of one vector, the second vector (of correct length and direction) is drawn; from its tip, the third vector is drawn; and so on, as illustrated in Fig. 2–7. The *vector sum* is the vector that extends *from* the beginning of the first vector *to* the tip of the last vector, i.e., the vector ($\mathbf{A} + \mathbf{B} + \mathbf{C}$ in the figure) which completes or "closes" the polygon. Note that this could be a twisted polygon in three-dimensional space; i.e., not in a plane.

Referring to Fig. 2–8, we see that $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$ is a vector in the same direction as $\mathbf{A}$ and of twice the length. Likewise $k\mathbf{A}$ is k times as long as $\mathbf{A}$ and in the same direction if k is positive. However, if k is negative, $k\mathbf{A}$ is in the opposite direction from $\mathbf{A}$ and is $|k|$ times as long. As noted above, k is called a scalar. If two vectors are parallel, one is a scalar times the other; and conversely, if $n\mathbf{A} = m\mathbf{B}$ (m and n being scalars not equal to zero), then $\mathbf{A}$ and $\mathbf{B}$ are parallel vectors. Further, $\mathbf{A}/|\mathbf{A}| = \mathbf{i}'$ is the unit vector (length $= 1$) in the *same direction* as $\mathbf{A}$, and $\mathbf{A} = |\mathbf{A}|\mathbf{i}'$.

(c) *Definition of subtraction.* $\mathbf{A} - \mathbf{B}$ is the vector which added to $\mathbf{B}$ gives vector $\mathbf{A}$. It is therefore the vector from the tip of $\mathbf{B}$ to the tip of $\mathbf{A}$. (See Fig. 2–9.) One could, of course, add $-\mathbf{B}$ to $\mathbf{A}$ to give $\mathbf{A} - \mathbf{B}$, as

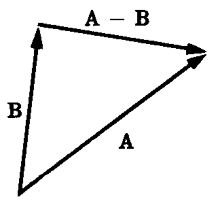


FIGURE 2–9

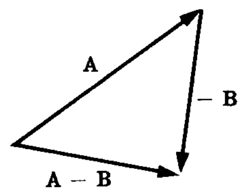


FIGURE 2–10

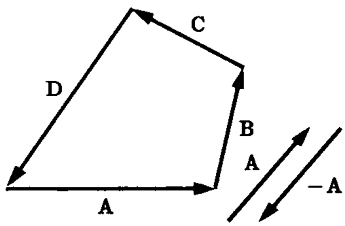


FIGURE 2–11

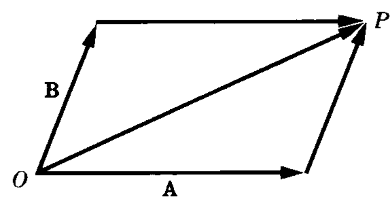


FIGURE 2–12

shown in Fig. 2–10. The second method is better for problems like $3\mathbf{A} - 2\mathbf{B} + 4\mathbf{C} - 2\mathbf{D}$.

Exercise: Prove that the two vectors given by these two methods, both alleged to be $\mathbf{A} - \mathbf{B}$, are actually the same (i.e., have the same length and same direction). See Figs. 2–9 and 2–10.

The operations of addition and subtraction lead to an important special vector. In the case where the addition of the vectors leads to a *closed* polygon, the endpoint of the last vector coinciding with the beginning point of the first vector, as in Fig. 2–11, we define the sum to be a vector of zero length, called the *null vector*, and we write $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$. The invention of the null vector makes addition of vectors always possible. The null vector does not have direction; it is undefined. Thus $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$, the null vector; also, $\mathbf{0} + \mathbf{A} = \mathbf{A} + \mathbf{0} = \mathbf{A}$; and if

$$m\mathbf{A} + n\mathbf{B} = \mathbf{0},$$

we can say that

$$n\mathbf{B} = \mathbf{0} - m\mathbf{A} = -m\mathbf{A}$$

(the vector which added to $m\mathbf{A}$ gives $\mathbf{0}$) by the definition of subtraction. This justifies "transposing" a vector from one side of an equation to the other, as in ordinary algebra.

One other remark should be made. Many readers may have already met addition of vectors in earlier work, using the "parallelogram law" of addition to find the "resultant" of two forces. The method is as follows: Two vectors $\mathbf{A}$ and $\mathbf{B}$ emanate from a common point O, as shown in Fig. 2–12. Then from the endpoint of each vector, a line is drawn parallel

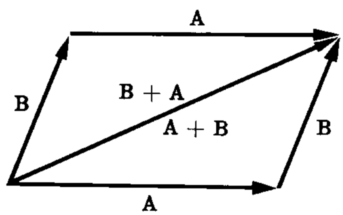


FIGURE 2–13

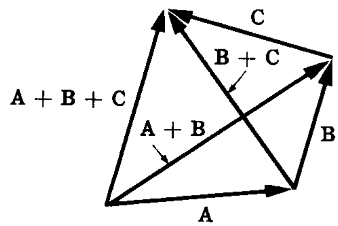


FIGURE 2–14

to the other vector, these lines intersecting at point P. Thus a parallelogram is formed. The diagonal vector $\overrightarrow{OP}$ is called the *resultant* or *sum* of the vectors **A** and **B**. This method agrees with the polygon method, since OP divides the parallelogram into two congruent triangles, either of which considered by itself is a special case of our polygon method of addition. The polygon rule is more general in that any number of vectors can be added at once, while by the parallelogram law a new parallelogram, formed by the preceding resultant with one additional vector, must be painstakingly drawn after each addition.

It is fairly evident from the definition of addition of vectors that:

(1) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (called the Commutative Law for Addition).

(2) $\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
$= (\mathbf{A} + \mathbf{B} + \mathbf{C})$ (the Associative Law for Addition).

(3) $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 \cdots + \mathbf{A}_n$ can be added in any order; for example, it is equal to $\mathbf{A}_3 + \mathbf{A}_1 + \mathbf{A}_6 \cdots + \mathbf{A}_2 + \cdots$.

The Commutative Law follows from the fact that $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$, emanating from a common point, form a parallelogram; and the diagonal thereof, as shown in Fig. 2–13, is simultaneously the vector $\mathbf{A} + \mathbf{B}$ and the vector $\mathbf{B} + \mathbf{A}$ (the Parallelogram Law again).

The Associative Law follows directly from the fact that when vectors are added, the same vector closes all three polygons, as shown in Fig. 2–14.

As to property (3), which is a generalized commutative law, our method of addition enables us to say that

$$\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 \cdots + \mathbf{A}_n$$

is the same no matter what set or sets of **A**'s are enclosed in parentheses. For example, we can write

$$(\mathbf{A}_1 + \mathbf{A}_2) + (\mathbf{A}_3 + \mathbf{A}_4 + \mathbf{A}_5) + (\mathbf{A}_k + \cdots \mathbf{A}_n),$$

a generalized associative law, since the vector necessary to close the polygon is always the same. Hence by placing parentheses around *pairs* and applying the Commutative Law repeatedly, we can put the **A**'s in any order we please.