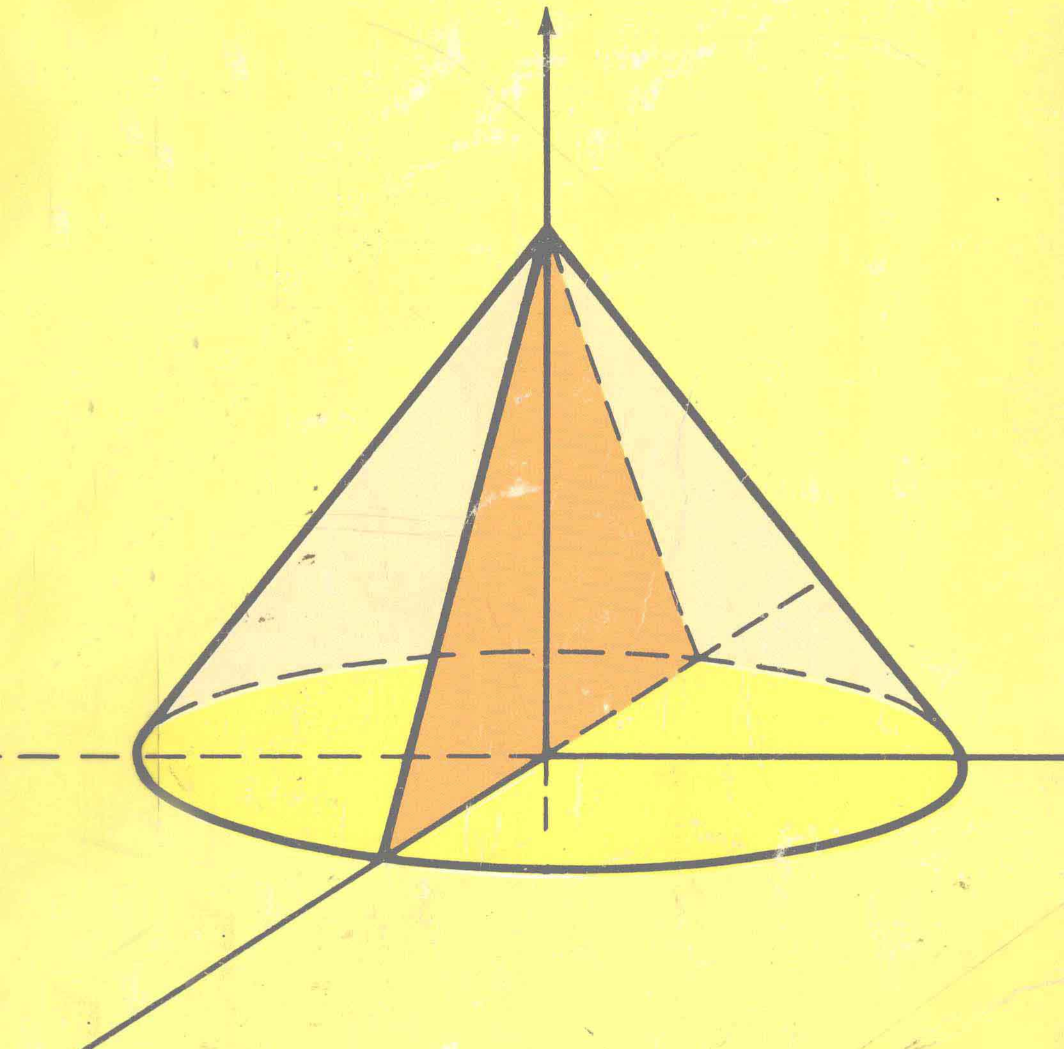
ESSENTIAL CALCULUS WITH APPLICATIONS



Richard A. Silverman

ESSENTIAL CALCULUS

with Applications

RICHARD A. SILVERMAN

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TO THE INSTRUCTOR

The attributes and philosophy of this book are best described by giving a running synopsis of each of the six chapters. This summary is accompanied by open expressions of my pedagogical preferences. Like most authors, I tend to regard these not as idiosyncrasies, but as the only reasonable way to do things! If you disagree in spots, I hope you will attribute this lack of modesty to an excess of enthusiasm, an occupational hazard of those with the effrontery to write books.

The contents of Chapter 1 are often called "precalculus," and are in fact just what that term implies, namely, material that ought to be at one's mathematical fingertips before attempting the study of calculus proper. Opinions differ as to what such a background chapter should contain. Some authors cannot wait to get on with the main show, even at the risk of talking about derivatives to students who are still struggling with straight lines, while others seem unwilling to venture into the heartland of calculus without a year's supply of mathematical rations. I have tried to strike a happy medium by travelling light, but well-equipped. Thus there is a brief section on sets, a larger one on numbers, a little bit on mathematical induction, and quite a lot on inequalities and absolute values, two topics that always seem to give students trouble despite their precalculus character. There is a whole section on intervals, both finite and infinite. The last three sections of the chapter administer a modest dose of analytic geometry, with the emphasis on straight lines and their equations. It should not take long to bring all the students up to the mathematical level of Chapter 1, regardless of their starting points, and those few who are there already can spend their spare time solving extra problems while the others catch up!

The class is now ready to attack Chapter 2, and with it the study of differential calculus. The chapter begins with a rather leisurely and entirely concrete discussion of the function concept. It is my belief that many books adopt too abstract an approach to this important subject. Thus I do not hesitate to use terms like "variable" and "argument," which some may regard as old-fashioned, relegating the mapping and ordered pair definitions of function to the problems. At the same time, I find this a natural juncture to say a few words about functions of several variables. After all, why should one have to wait until the very end of the book to write a simple equation like $F(x, y) = 0$? And what's wrong with a few examples of nonnumerical functions, which crop up all the time in the social sciences? While still in the first three sections of Chapter 2, the student encounters one-to-one functions and inverse functions, and then composite functions and sequences after specializing to numerical functions of a single variable. Graphs of equations and functions are treated in terms of solution sets, with due regard for parity of functions and its consequences for the symmetry of their graphs.

Having mastered the concept of function, in all its various manifestations, the student now arrives at Sec. 2.4, where derivatives and limits are introduced *simultaneously*. I am of the opinion that the novice can hardly develop any respect for the machinery of limits, without first being told that limits are needed to define *derivatives*. Here the development of the individual's understanding must recapitulate the actual historical evolution of the subject. For the same reason, I feel that no time should be wasted in getting down to such brass tacks as difference quotients, rates of change, and increments. Moreover, after defining the tangent to a curve, I find it desirable to immediately say something about differentials. This is a small price to pay for the ability to motivate the ubiquitous "*d* notation," and differentials have many other uses too (for example, in Secs. 4.6 and 6.2).

It is now time for the student to learn more about limits. This is done in Sec. 2.6, where a number of topics are presented in quick order, namely, algebraic operations on limits, one-sided limits, the key concept of continuity, algebraic operations on continuous functions, and the fact that differentiability implies continuity. Armed with this information, one can now become a minor expert on differentiation, by mastering the material in Secs. 2.7 and 2.8. After establishing the basic differentiation formula $(x^r)' = rx^{r-1}$ for r a positive or negative integer, I authorize the student to make free use of the same formula for r an arbitrary real number. Why waste time justifying special cases when the "master formula" itself will be proved once and for all in Sec. 4.4? (However, in a concession to tradition, the validity of the formula for r a rational number is established in the problems, in the usual two ways.) Following a brief discussion of higher derivatives, the student arrives next at the rule for differentiating an inverse function and the all-important chain rule. Unlike most authors, I use a proof of the chain rule which completely avoids the spurious difficulty stemming from the possibility of a vanishing denominator, and which has the additional merit of generalizing at once to the case of functions of several variables (see Sec. 6.3). The method of implicit differentiation is treated as a corollary of the chain rule, and I do not neglect to discuss what can go wrong with the method if it is applied blindly. Chapter 2, admittedly a long one, closes with a comprehensive but concise treatment of limits of other kinds, namely, limits involving infinity, asymptotes, the limit of an infinite sequence, and the sum of an infinite series. Once having grasped the concept of the limit of a function at a point, the student should have little further difficulty in assimilating these variants of the limit concept, and this seems to me the logical place to introduce them.

In Chapter 3 differentiation is used as a tool, and the book takes a more practical turn. I feel that the concept of velocity merits a section of its own, as do related rates and the concept of marginality in economic theory. It is then time to say more about the properties of continuous functions and of differentiable functions, and I do so in that order since the student is by now well aware that continuity is a weaker requirement than differentiability. The highly plausible fact that a continuous image of a closed interval is itself a closed interval leads to a quick proof of the existence of global extrema for a continuous function defined in a closed interval, with the intermediate value theorem as an immediate consequence. The connection between the sign of the derivative of a function at a point and its behavior in a neighborhood of the point is then used to prove Rolle's theorem and the mean value theorem, in turn. With the mean value theorem now available, I immediately exploit the opportunity to introduce the antiderivative and the indefinite integral, which will soon be needed to do integral calculus.

The chapter goes on to treat local extrema, including the case where the function under investigation may fail to be differentiable at certain points. Both the first and second derivative tests for a strict local extremum are proved in a straightforward way, with the help of the mean value theorem. The next section, on concavity and inflection points, is somewhat of an innovation, in that it develops a complete parallelism between the theory of monotonic functions and critical points, on the one hand, and the theory of concave functions and inflection points, on the other. The chapter ends with a discussion of concrete optimization problems, and the three solved examples in Sec. 3.8 are deliberately chosen to be nontrivial, so that the student can have a taste of the "real thing."

It is now Chapter 4, and high time for integral calculus. Here I prefer to use the standard definition of the Riemann integral, allowing the points ξ_i figuring in the approximating sum σ to be *arbitrary* points of their respective subintervals. Students seem to find this definition perfectly plausible, in view of the interpretation of σ as an approximation to the area under the graph of the given function. Once the definite integral is defined, it is immediately emphasized that all continuous functions are integrable, and this fact is henceforth used freely. After establishing a few elementary properties of definite integrals, I prove the mean value theorem for integrals and interpret it geometrically. It is then a simple matter to prove the fundamental theorem of calculus. Next the function $\ln x$ is defined as an integral, in the usual way, and its properties and those of its inverse function e^x are systematically explored. The related functions $\log_a x$, a^x and x^r are treated on the spot, and the validity of the formula $(x^r)' = rx^{r-1}$ for arbitrary real r is finally proved, as promised back in Chapter 2. The two main techniques of integration, namely, integration by substitution and integration by parts, are discussed in detail. The chapter ends with a treatment of improper integrals, both those in which the interval of integration is infinite and those in which the integrand becomes infinite.

There are various ways in which integration can be used as a tool, but foremost among these is certainly the use of integration to solve differential equations. It is for this reason that I have made Chapter 5 into a brief introduction to differential equations and their applications. All the theory needed for our purposes is developed in Sec. 5.1, both for first-order and second-order equations. The next section is then devoted to problems of growth and decay, a subject governed by simple first-order differential equations. The standard examples of population growth, both unrestricted and restricted, are gone into in some detail, as is the topic of radioactive decay. The last section of this short chapter is devoted to problems of motion, where second-order differential equations now hold sway. Inclusion of this material may be regarded as controversial in a book like this, but I for one do not see anything unreasonable in asking even a business or economics student to devote a few hours to the contemplation of Newton's mechanics, a thought system which gave birth first to modern industrial society and then to the space age. In any event, those who for one reason or another still wish to skip Sec. 5.3 hardly need my permission to do so.

The last of the six chapters of this book is devoted to the differential calculus of functions of several variables. Here my intent is to highlight the similarities with the one-dimensional case, while not neglecting significant differences. For example, this is why I feel compelled to say a few words about the distinction between differentiable functions of several variables and those that merely have partial derivatives. However, I do not dwell on such matters. It turns out that much of the theory of Chapters 2 and 3 can be generalized almost effortlessly to the n-dimensional case, without doing violence to the elementary character of the book. In particular, as already noted, the proof of the chain rule in Sec. 6.3 is virtually the same as the one in Sec. 2.8. Chapter 6 closes with a concise treatment of extrema in n dimensions, including the test for strict local extrema and the use of Lagrange multipliers to solve optimization problems subject to constraints. I stop here, because unlike some authors I see no point in reproducing the standard examples involving indifference curves, budget lines, marginal rates of substitution, and the like, to be found in every book on microeconomic theory. I conceive of this book as one dealing primarily with the common mathematical ground on which many subjects rest, and the applications chosen here are ones which shed most light on the kind of mathematics we are trying to do, not those which are most intriguing from other points of view.

The idea of writing this book in the first place was proposed to me by John S. Snyder, Jr. of the W. B. Saunders Co. Without his abiding concern, I find it hard to imagine that the book would ever have arrived at its present form. In accomplishing a total overhaul of an earlier draft, I was guided by helpful suggestions from a whole battery of reviewers, notably, Craig Comstock of the Naval Postgraduate School, John A. Pfaltzgraff of the University of North Carolina, J. H. Curtiss of the University of Miami, Carl M. Bruns of Florissant Valley Community College, David Brown of the University of Pittsburgh, and Maurice Beren of the Lowell Technological Institute. The last of these reviewers played a particularly significant role in my revision of Chapter 1. I would also like to thank my friend Neal Zierler for checking all the answers to the problems in the first draft of the book, and my copy-editor Lloyd Black for his patience in dealing with the kind of author who keeps reading proof, looking for trouble, until it is finally taken away from him once and for all. It has been a pleasure to work with all these fine people.

TO THE STUDENT

Calculus cannot be learned without solving lots of problems. Your instructor will undoubtedly assign you many problems as homework, probably from among those that do not appear in the Selected Hints and Answers section at the end of the book. But, at the same time, every hint or answer in that section challenges you to solve the corresponding problem, whether it has been assigned or not. This is the only way that you can be sure of your command of the subject. Problems marked with stars are either a bit harder than the others, or else they deal with side issues. However, there is no reason to shun these problems. They're neither that hard nor that far off the main track.

The system of cross references used in this book is almost self-explanatory. For example, Theorem 1.48 refers to the one and only theorem in Sec. 1.48, Example 2.43b refers to the one and only example in Sec. 2.43b, and so on. Any problem cited without a further address will be found at the end of the section where it is mentioned. The book has a particularly complete index to help you find your way around. Use it freely.

Mathematics books are not novels, and you will often have to read the same passage over and over again before you grasp its meaning. Don't let this discourage you. With a little patience and fortitude, you too will be doing calculus before long. Good luck!

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Chapter 1

MATHEMATICAL BACKGROUND

1.1 INTRODUCTORY REMARKS

1.11. You are about to begin the study of calculus, a branch of mathematics which dates back to the seventeenth century, when it was invented by Newton and Leibniz independently and more or less simultaneously. At first, you will be exposed to ideas that you may find strange and abstract, and that may not seem to have very much to do with the "real world." After a while, though, more and more applications of these ideas will put in an appearance, until you finally come to appreciate just how powerful a tool calculus is for solving a host of practical problems in fields as diverse as physics, biology and economics, just to mention a few.

Why this delay? Why can't we just jump in feet first, and start solving practical problems right away? Why must the initial steps be so methodical and careful?

The reason is not hard to find, and it is a good one. You are in effect learning a new language, and you must know the meaning of key words and terms before trying to write your first story in this language, that is, before solving your first nonroutine problem. Many of the concepts of calculus are unfamiliar, and were introduced, somewhat reluctantly, only after it gradually dawned on mathematicians that they were in fact indispensable. This is certainly true of the central concept of calculus, namely, the notion of a "limit," which has been fully understood only for a hundred years or so, after having eluded mathematicians for millennia. Living as we do in the modern computer age, we can hardly expect to learn calculus in archaic languages, like that of "infinitesimals," once so popular. We must also build up a certain amount of computational facility, especially as involves *inequalities*, before we are equipped to tackle the more exciting problems of calculus. And we must become accustomed to think both algebraically and geometrically at the same time, with the help of rectangular coordinate systems. All this "tooling up" takes time, but nowhere near as much as in other fields, like music, with its endless scales and exercises. After all, in calculus we need only train our minds, not our hands!

It is also necessary to maintain a certain generality in the beginning, especially in connection with the notion of a "function." The power of calculus is intimately related to its great generality. This is why so many different kinds of problems can be solved by the methods of calculus. For example, calculus deals with "rates of change" in general, and not just special kinds of rates of change, like "velocity," "marginal cost" and "rate of cooling," to mention only three. From the calculus point of view, there are often deep similarities between things that appear superficially unrelated.

In working through this book, you must always have your pen and scratch pad at your side, prepared to make a little calculation or draw a rough figure at a moment's notice. Never go on to a new idea without understanding the old ideas on which it is based. For example, don't try to do problems involving "continuity" without having mastered the idea of a "limit." This is really a workshop course, and your only objective is to learn how to solve calculus problems. Think of an art class, where there is no premium on anything except making good drawings. That will put you in the right frame of mind from the start.

1.12. Two key problems. Broadly speaking, calculus is the mathematics of change. Among the many problems it deals with, two play a particularly prominent role, in ways that will become clearer to you the more calculus you learn. One problem is

(1) Given a relationship between two changing quantities, what is the rate of change of one quantity with respect to the other?

And the other, so-called "converse" problem is

(2) Given the rate of change of one quantity with respect to another, what is the relationship between the two quantities?

Thus, from the very outset, we must develop a language in which "relationships," whatever they are, can be expressed precisely, and in which "rates of change" can be defined and calculated. This leads us straight to the basic notions of "function" and "derivative." In the same way, the second problem leads us to the equally basic notions of "integral" and "differential equation." It is the last concept, of an equation involving "rates of change," that unleashes the full power of calculus. You might think of it as "Newton's breakthrough," which enabled him to derive the laws of planetary motion from a simple differential equation involving the force of gravitation. Why *does* an apple fall?

We will get to most of these matters with all deliberate speed. But we must first spend a few sections reviewing that part of elementary mathematics which is an indispensable background to calculus. Admittedly, this is not the glamorous part of our subject, but first things first! We must all stand on some common ground. Let us begin, then, from a starting point where nothing is assumed other than some elementary algebra and geometry, and a little patience.

1.2 SETS

A little set language goes a long way in simplifying the study of calculus. However, like many good things, sets should be used sparingly and only when the occasion really calls for them.

1.21. A collection of objects of any kind is called a *set*, and the objects themselves are called *elements* of the set. In mathematics the elements are usually numbers or symbols. Sets are often denoted by capital letters and their elements by small letters. If x is an element of a set A, we may write $x \in A$, where the symbol $\in$ is read "is an element of." Other ways of reading $x \in A$ are "x is a member of A," "x belongs to A," and "A contains x." For example, the set of all Portuguese-speaking countries in Latin America contains a single element, namely Brazil.

1.22. If every element of a set A is also an element of a set B, we write $A \subset B$, which reads "A is a *subset* of B." If A is a subset of B, but B is not a subset of A, we say that A is a *proper subset* of B. In simple language, this means that B not only contains all the elements of A, but also one or more extra elements. For example, the set of all U.S. Senators is a proper subset of the set of all members of the U.S. Congress.

1.23. a. One way of describing a set is to write its elements between curly brackets. Thus the set $\{a, b, c\}$ is made up of the elements a, b and c. Changing the order of the elements does *not* change the set. For example, the set $\{b, c, a\}$ is the same as $\{a, b, c\}$. Repeating an element does not change a set. For example, the set $\{a, a, b, c, c\}$ is the same as $\{a, b, c\}$.

b. We can also describe a set by giving properties that uniquely determine its elements, often using the colon: as an abbreviation for the words "such that." For example, the set $\{x: x = x^2\}$ is the set of all numbers x which equal their own squares. You can easily convince yourself that this set contains only two elements, namely 0 and 1.

1.24. Union of two sets. The set of all elements belonging to at least one of two given sets A and B is called the *union* of A and B. In other words, the union of A and B is made up of all the elements which are in the set A or in the set B, or possibly in both. We write the union of A and B as $A \cup B$, which is often read "A cup B," because of the shape of the symbol $\cup$. For example, if A is the set $\{a, b, c\}$ and B is the set $\{c, d, e\}$, then $A \cup B$ is the set $\{a, b, c, d, e\}$.

1.25. Intersection of two sets. The set of all elements belonging to both of two given sets A and B is called the *intersection* of A and B. In other words, the intersection of A and B is made up of only those elements of the sets A and B which are in both sets; elements which belong to only one of the sets A and B do not belong to the intersection of A and B. We write the intersection of A and B as $A \cap B$, which is often read "A cap B," because of the shape of the symbol $\cap$. For example, if A is the set $\{a, b, c, d\}$ and B is the set $\{b, d, e, f, g\}$, then $A \cap B$ is the set $\{b, d\}$.

1.26. Empty sets. A set which has no elements at all is said to be an *empty set* and is denoted by the symbol $\varnothing$. For example, the set of unicorns in the Bronx Zoo is empty.

By definition, an empty set is considered to be a subset of every set. This is just a mathematical convenience.

1.27. Equality of sets. We say that two sets A and B are *equal* and we write $A = B$ if A and B have the same elements. If A is empty, we write $A = \varnothing$. For example, $\{x: x = x^2\} = \{0, 1\}$, as already noted, while $\{x: x \neq x\} = \varnothing$ since no number x fails to equal itself!

PROBLEMS

1. Find all the proper subsets of the set $\{a, b, c\}$.
2. Write each of the following sets in another way, by listing elements:
   (a) $\{x: x = -x\}$; (b) $\{x: x + 3 = 8\}$; (c) $\{x: x^2 = 9\}$;
   (d) $\{x: x^2 - 5x + 6 = 0\}$; (e) $\{x: x$ is a letter in the word "calculus"$\}$.
3. Let $A = \{1, 2, \{3\}, \{4, 5\}\}$. Which of the following are true?
   (a) $1 \in A$; (b) $3 \in A$; (c) $\{2\} \in A$.
   How many elements does A have?
4. Which of the following are true?
   (a) If $A = B$, then $A \subset B$ and $B \subset A$; (b) If $A \subset B$ and $B \subset A$, then $A = B$;
   (c) $\{x: x \in A\} = A$; (d) $\{$all men over 80 years old$\} = \varnothing$.
5. Find the union of the sets A and B if
   (a) $A = \{a, b, c\}$, $B = \{a, b, c, d\}$; (b) $A = \{1, 2, 3, 4\}$, $B = \{-1, 0, 2, 3\}$.
6. Find the intersection of the sets A and B if
   (a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$; (b) $A = \{a, b, c, d\}$, $B = \{f, g, h\}$.
7. Given any set A, verify that $A \cup A = A \cap A = A$.
8. Given any two sets A and B, verify that both A and B are subsets of $A \cup B$, while $A \cap B$ is a subset of both A and B.
9. Given any two sets A and B, verify that $A \cap B$ is always a subset of $A \cup B$. Can $A \cap B$ ever equal $A \cup B$?
10. Given any two sets A and B, by the *difference* $A - B$ we mean the set of all elements which belong to A but not to B. Let $A = \{1, 2, 3\}$. Find $A - B$ if
    (a) $B = \{1, 2\}$; (b) $B = \{4, 5\}$; (c) $B = \varnothing$; (d) $B = \{1, 2, 3\}$.
11. Which of the following sets are empty?
    (a) $\{x: x$ is a letter before c in the alphabet$\}$;
    (b) $\{x: x$ is a letter after z in the alphabet$\}$;
    (c) $\{x: x + 7 = 7\}$;
    (d) $\{x: x^2 = 9$ and $2x = 4\}$.

*12. Which of the following sets are empty?
   (a) The set of all right triangles whose side lengths are whole numbers;
   (b) The set of all right triangles with side lengths in the ratio 5:12:13;
   (c) The set of all regular polygons with an interior angle of 45 degrees;
   (d) The set of all regular polygons with an interior angle of 90 degrees;
   (e) The set of all regular polygons with an interior angle of 100 degrees.
   Explain your reasoning.

   Comment. A polygon is said to be *regular* if all its sides have the same length and all its interior angles are equal.

*13. Let $A = \{a, b, c, d\}$, and let B be the set of all subsets of A. How many elements does B have?

1.3 NUMBERS

In this section we discuss numbers of various kinds, beginning with integers and rational numbers and moving on to irrational numbers and real numbers. The set of all real numbers is called the *real number system.* It is the number system needed to carry out the calculations called for in calculus.

1.31. The number line. Suppose we construct a straight line L through a point

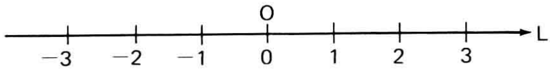

Figure 1.

O and extend it indefinitely in both directions. Selecting an arbitrary unit of measurement, we mark off on the line to the right of O first 1 unit, then 2 units, 3 units, and so on. Next we do the same thing to the left of O. The marks to the right of O correspond to the *positive integers* 1, 2, 3, and so on, and the marks to the left of O correspond to the *negative integers* -1, -2, -3, and so on. The line L, "calibrated" by these marks, is called the *number line*, and the point O is called the *origin* (of L). The direction from negative to positive numbers along L is called the *positive direction*, and is indicated by the arrowhead in Figure 1.

1.32. Integers

a. The set of positive integers is said to be *closed* under the operations of addition and multiplication. In simple language, this means that if we add or multiply two positive integers, we always get another positive integer. For example, $2 + 3 = 5$ and $2 \cdot 3 = 6$, where 5 and 6 are positive integers. On the other hand, the set of positive integers is *not* closed under subtraction. For example, $2 - 3 = -1$, where -1 is a negative integer, rather than a positive integer.

The number 0 corresponding to the point O in Figure 1 is called *zero*. It can be regarded as an integer which is neither positive nor negative. Following mathematical tradition, we use the letter Z to denote the set of all integers, positive, negative and zero. The set Z, unlike the set of positive integers, is closed under subtraction. For example, $4 - 2 = 2$, $3 - 3 = 0$ and $2 - 5 = -3$, where the numbers 2, 0 and -3 are all integers, whether positive, negative or zero.

b. An integer n is said to be an *even number* if $n = 2k$, where k is another integer, that is, if n is divisible by 2. On the other hand, an integer n is said to be an *odd number* if $n = 2k + 1$, where k is another integer, that is, if n is not divisible by 2, or equivalently leaves the remainder 1 when divided by 2. It is clear that every integer is either an even number or an odd number.

1.33. Rational numbers. The set Z is still too small from the standpoint of someone who wants to be able to *divide* any number in Z by any other number in Z and still be sure of getting a number in Z. In other words, the set Z is not closed under division. For example, $2 \div 3 = \frac{2}{3}$ and $-4 \div 3 = -\frac{4}{3}$, where $\frac{2}{3}$ and $-\frac{4}{3}$ are fractions, not integers. Of course, the quotient of two integers is *sometimes* an integer, and this fact is a major preoccupation of the branch of mathematics known as *number theory*. For example, $8 \div 4 = 2$ and $10 \div -5 = -2$. However, to make division possible in general, we need a bigger set of numbers than Z. Thus we introduce *rational numbers*, namely fractions of the form m/n, where m and n are both integers and *n is not zero*. Note that every integer m, including zero, is a rational number, since $m/1 = m$.

Let Q (for "quotient") denote the set of all rational numbers. Then the set Q is closed under the four basic arithmetical operations of addition, subtraction, multiplication and division, *provided that we never divide by zero*. It cannot be emphasized too strongly that *division by zero is a forbidden operation* in this course. These matters are considered further in Problems 3 and 13.

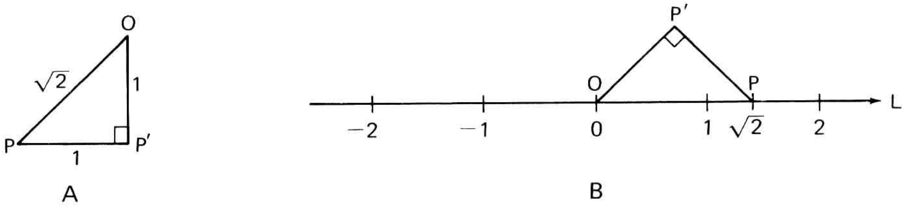

Figure 2.

1.34. Irrational numbers

a. With respect to the number line, the rational numbers fill up the points corresponding to the integers and many but *not all* of the points in between. In other words, there are points of the number line which do *not* correspond to rational numbers. To see this, suppose we construct a right triangle $PP'O$ with sides PP' and $P'O$ of length 1, as in Figure 2A. Then, by elementary geometry, the side OP is of length $\sqrt{1^2 + 1^2} = \sqrt{2}$ (use the familiar Pythagorean theorem). Suppose we place the side OP on the number line, as in Figure 2B, with the point O coinciding with the origin of the line. Then the point P corresponds to the number $\sqrt{2}$. But, as mathematicians concluded long ago, the number $\sqrt{2}$ cannot be rational, and therefore P is a point of the number line which does not correspond to a rational number.

b. By an *irrational number* we simply mean a number, like $\sqrt{2}$, which is not rational. To demonstrate that $\sqrt{2}$ is irrational, we argue as follows. First we digress for a moment to show that the result of squaring an odd number (Sec. 1.32b) is always an odd number. In fact, every odd number is of the form $2k + 1$, where k is an integer, and, conversely, every number of this form is odd. But, squaring the expression $2k + 1$, we get

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd, since $2k^2 + 2k$ is itself an integer (why?).

Now, returning to the main argument, suppose $\sqrt{2}$ is a rational number. Then $\sqrt{2}$ must be of the form m/n, where m and n are positive integers and we can assume that the fraction m/n has been reduced to lowest terms, so that m and n are no longer divisible by a common factor other than 1. (For example, the fraction $\frac{12}{18}$ is not in lowest terms, but the equivalent fraction $\frac{2}{3}$ is.) We can then write

$$\sqrt{2} = \frac{m}{n}. \tag{1}$$

Squaring both sides of (1), we have

$$2 = \frac{m^2}{n^2},$$

or equivalently

$$m^2 = 2n^2. \tag{2}$$

Thus m^2 is an even number, being divisible by 2, and therefore the number m itself must be even, since if m were odd, m^2 would also be odd, as shown in the preceding paragraph. Since m is even, we can write m in the form

$$m = 2k, \tag{3}$$