Plane Trigonometry

second edition

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Metric and English Measurements

Metric Equivalents		English Equivalents
	Length	
1 meter = 1000 millimeters = 100 centimeters = 0.001 kilometer		1 foot = 12 inches 1 yard = 3 feet
	Area	
1 hectare = 10,000 sq. meters		1 acre = 43,560 sq. feet
	Volume	
1 liter = 1000 milliliters = 0.001 kiloliter		1 quart = 2 pints
	Weight	
1 gram = 1000 milligrams = 0.001 kilogram		1 pound = 16 ounces

Approximate Equivalents

Metric–English	English–Metric
1 centimeter = 0.3937 inch	1 inch = 2.54 centimeters
1 meter = 39.37 inches	1 yard = 0.914 meter
1 meter = 3.281 feet	1 foot = 0.3048 meter
1 kilometer = 0.62 mile	1 mile = 1.6 kilometers
1 liter = 1.057 U.S. quarts	1 quart = 0.946 liter
1 kilogram = 2.2 pounds	1 pound = 0.45 kilogram
1 hectare = 2.47 acres	1 acre = .404 hectare

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Preface

In this edition of *Plane Trigonometry* we have added many new example problems and exercises and have attempted to polish some of the rough edges that occur in most first editions. However, our basic approach to trigonometry remains unchanged. We believe that trigonometry is best learned, best understood, and best remembered with respect to its right triangle definitions extended to angles in standard position on the rectangular Cartesian coordinate system. Our initial use of the more classical definitions of the trigonometric ratios combined with a modern flavor in the applications provides the student with an understanding of trigonometry that cannot be obtained by taking an entirely analytic approach to the subject.

The modern analytical aspects of trigonometry are far from ignored; most of the material in Chapters 5–8 is devoted to analytic trigonometry and related nontriangular applications. Set notation is used whenever appropriate and heavy emphasis is placed on graphing so that the student can "see" solution sets as well as derive them.

As with most books, there are more topics included than most instructors would cover in a normal one-semester or one-quarter course. This book is constructed so that topics can frequently be omitted without losing continuity. For instance, Chapters 2 and 4 include a large variety of practical applications allowing the individual instructor a wide range of personal choice. Chapter 9 on logarithms and Chapter 10 on complex numbers are dependent on earlier chapters, but complex numbers themselves are not used elsewhere in the book.

Chapter 1 is a review of those areas considered to be fundamental to understanding the concepts of trigonometry. Some classes will be able to skim this chapter or skip it completely; others will need to spend more time if they are deficient in these areas. In either case, the chapter is a convenient reference for the student. We have found it useful to spend a day or two reviewing the notation and facts about angles presented in this chapter before proceeding to Chapter 2.

Several different types of courses are possible using this book. For a course that is triangle-oriented, we suggest the following:

(1) Review topics in Chapter 1 as needed
(2) Sections 2.1–2.6 complete
(3) Selected topics from Sections 2.7–2.9 as time permits
(4) Sections 3.1, 3.2, 3.3 (optional), and 3.4
(5) Sections 4.1–4.3
(6) Sections 4.4–4.6 as time permits
(7) Section 5.1
(8) Sections 6.1–6.3
(9) Sections 7.1, 7.2, and 7.3
(10) Sections 8.1 and 8.3

For a course with a more analytical approach:

(1) Review topics in Chapter 1 as needed
(2) Sections 2.1–2.6
(3) Sections 3.1, 3.2, and 3.4
(4) Chapters 5, 6, 7, and 8
(5) Sections 4.2 and 4.3 as time permits

In either of these two courses optional topics such as logarithms, complex numbers, and polar coordinates can be added as needed. The applications are more than plentiful and some of them must of course be omitted in any one particular course.

In this edition we have given added attention to applications, both triangular and analytical. Hundreds of exercises have been added and many new worked-out examples have been incorporated into the text. Chapter tests have been added to give the instructor an additional source for review.

Once again we wish to thank Mr. Myrl H. Ahrendt and the National Aeronautics and Space Administration for their kind permission to use some of the unique applications of trigonometry to space technology from the book *Space Mathematics: A Resource for Teachers.*

We also wish to take this opportunity to thank Mr. John R. Nash and Mr. John Xanthos, who have been of great assistance to us in working out the answers. The correctness of the answers is, of course, our responsibility, but they have greatly contributed to their improvement. In this regard we have found that the accuracy of an answer is often dependent on whether a calculator or the tables in the book are used and which truncated approximations are used in the operations. We are aware of the disconcerting effect of such variations, but we know of no easy solution to this accuracy problem.

This edition of *Plane Trigonometry* has benefited from critical reviews and comments from the users of the first edition. We wish to single out the following people who assisted us with their valuable comments:

Bill D. Anderson, East Texas State University; Richard A. Askey, University of Wisconsin; Frank E. Burk, California State University, Chico; Robert Allan Chaffer, Central Michigan University; James L. Cornette, Iowa State University; Kenneth J. Davis, East Carolina University; Claude B. Duplissey, University of Arkansas at Little Rock; Steven C. Ferry, University of Kentucky; Gerald K. Goff, Oklahoma State University; Peter J. Nicholls, Northern Illinois University; Elizabeth A. Phillips, Michigan State University; Roberta L. White, Frostburg State University; Albert W. Zechmann, University of Nebraska, Lincoln.

Special thanks are due Professor Curtis A. Rogers, University of Houston, who, besides offering many useful suggestions in this edition, provided us with an extensive list of new exercises.

It is a pleasure to acknowledge the fine cooperation of the staff of Prindle, Weber & Schmidt, particularly our editor Mr. Locke Macdonald.

BERNARD J. RICE
JERRY D. STRANGE
University of Dayton
February 1978

Trigonometric Identities

(1) $\sin A = \dfrac{1}{\csc A}$

(2) $\cos A = \dfrac{1}{\sec A}$

(3) $\tan A = \dfrac{1}{\cot A}$

(4) $\tan A = \dfrac{\sin A}{\cos A}$

(5) $\cot A = \dfrac{\cos A}{\sin A}$

(6) $\sin^2 A + \cos^2 A = 1$

(7) $1 + \tan^2 A = \sec^2 A$

(8) $1 + \cot^2 A = \csc^2 A$

(9) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(10) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(11) $\cos(A + B) = \cos A \cos B + \sin A \sin B$

(12) $\cos(A - B) = \cos A \cos B - \sin A \sin B$

(13) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(14) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(15) $\sin 2A = 2 \sin A \cos A$

(16) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(17) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(18) $\sin \frac{1}{2} x = \pm\sqrt{(1 - \cos x)/2}$

(19) $\cos \frac{1}{2} x = \pm\sqrt{(1 + \cos x)/2}$

(20) $\tan \frac{1}{2} x = \dfrac{\sin x}{1 + \cos x}$

(21) $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

(22) $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

(23) $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

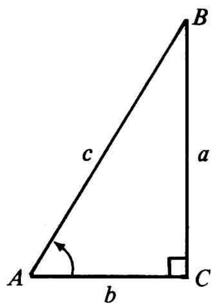
(24) $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

(25) $\sin A \cos B = \frac{1}{2}\{\sin(A + B) + \sin(A - B)\}$

(26) $\cos A \sin B = \frac{1}{2}\{\sin(A + B) - \sin(A - B)\}$

(27) $\cos A \cos B = \frac{1}{2}\{\cos(A + B) + \cos(A - B)\}$

Right Triangles

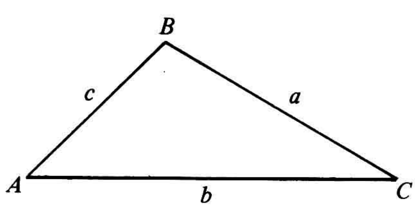


$$\sin\theta = \frac{a}{c} \qquad \cot\theta = \frac{b}{a}$$

$$\cos\theta = \frac{b}{c} \qquad \sec\theta = \frac{c}{b}$$

$$\tan\theta = \frac{a}{b} \qquad \csc\theta = \frac{c}{a}$$

Oblique Triangles



Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Contents

1
Some Fundamental Concepts

1.1 Historical Background

Trigonometry is one of the oldest branches of mathematics. An ancient scroll called the Ahmes Papyrus, written about 1550 B.C., contains problems that are solved by using similar triangles, the heart of the trigonometric idea. There is historical verification that measurements of distance and height were made by the Chinese about 1100 B.C. using what is essentially right triangle trigonometry. The subject eventually became intertwined with the study of astronomy. In fact, it is the Greek astronomer Hipparchus (180–125 B.C.) who is credited with compiling the first trigonometric tables and thus has earned the right to be known as "the father of trigonometry." The trigonometry of Hipparchus and the other astronomers was strictly a tool of measurement, and it is, therefore, difficult to refer to the early uses of the subject as either mathematics or astronomy.

In the 15th century, trigonometry was developed as a discipline within mathematics by Johann Muller, (1436–1476). This development created an interest in trigonometry throughout Europe and had the effect of placing Europe in a position of prominence with respect to astronomy and trigonometry.

In the 18th century, trigonometry was systematically developed in a completely different direction, highlighted by the publication in 1748 of the now famous "Introduction to Infinite Analysis" by Leonhard Euler (1707–1783). From this new viewpoint, trigonometry did not necessarily have to be considered in relation to a right triangle. Rather, the analytic or functional properties became paramount. As this wider outlook of the subject evolved, many new applications arose, especially as a tool for describing physical phenomena that are "periodic."

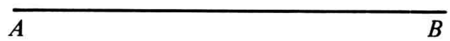
In this book we proceed more or less as the subject developed historically. First, we consider the trigonometry of a right triangle and only later introduce the analytic generalization that is so valuable in other areas of mathematics and physics.

To read the book profitably, you should have some ability with elementary algebra, particularly manipulative skills. Some of the specific background knowledge you will need is presented in this chapter.

1.2 Linear Measurement

In the early history of mathematics, as it is today, people were very much concerned with measuring distances of one kind or another. Technically, when measuring distances, we are actually measuring line segments. The actual measurement is called the **length** of the segment. Relatively small line segments are measured directly using a device such as a ruler or odometer. A line segment is designated by giving its end points, as in Figure 1.1, where the line segment is called AB. The length of such a line segment is variously denoted by $|AB|$, $\overline{AB}$, or $m(AB)$.

Figure 1.1 **Line segment *AB***



Units of length depend not only on the size of the segment being discussed but also on the system being used. For instance, in the English system, the fundamental unit of length is the foot, and in the International System, it is the meter.*

Trigonometry was originally developed as a tool for indirectly measuring the length of a line segment, that is, for determining the length of a line segment without using a physical measuring device. Of course, some measurements must be made but not necessarily of the line segment in question. For instance, the early astronomers could not measure the distance to the moon, but they could use trigonometry and a distance measured along the surface of the earth to compute the distance to the moon.

1.3 Angles and Their Measurement

When two line segments meet, they form what is called an **angle.** We ordinarily think of an angle as formed by two half-lines *OA* and *OB* that extend from some common point *O*, called the **vertex.** The half-lines are called the **sides** of the angle. (See Figure 1.2.)

We refer to an angle by mentioning a point on each of its sides and the vertex. Thus the angle in Figure 1.2 is called "the angle *AOB*," and is written $\angle AOB$. If there is only one angle under discussion whose vertex is at *O*, we sometimes simply say, "the angle at *O*," or more simply, "angle *O*." It is also customary to use Greek letters to designate angles. For example, $\angle AOB$ might also be called the angle θ (read "theta").

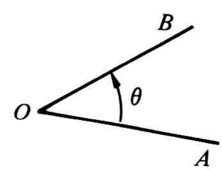


Figure 1.2

Often, in trigonometry, we must conceive of an angle as being "formed" by rotating one of the sides about its vertex while keeping the other side fixed as shown in Figure 1.3. If we think of *OA* as being fixed and *OB* as rotating about the vertex, *OA* is called the **initial side** and *OB* the **terminal side** of the generated angle. Other terminal sides such as OB' and OB'' result in different angles. The *size* of the angle depends on the amount of rotation of the terminal side. Thus, $\angle AOB$ is considered

*The International System of units (SI) replaces the metric system and is the system that the United States will adopt as we "go metric."

smaller than $\angle AOB'$ which, in turn, is smaller than $\angle AOB''$. Two angles are equal (in size) if they are formed by the same amount of rotation of the terminal side.

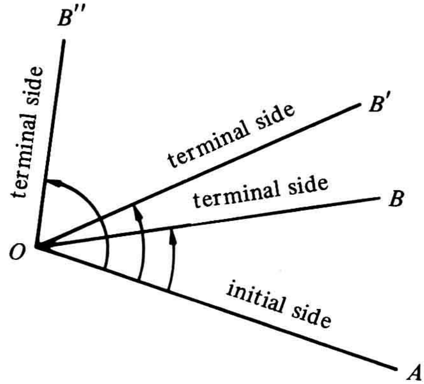


Figure 1.3

The Degree

The most commonly used unit of angular measure is the **degree.** We will take as definition that the measure of an angle formed by one complete revolution of the terminal side about its vertex is 360 degrees, also written 360°. One half of this angle, 180°, is called a **straight angle** and one fourth of it, 90°, is called a **right angle.** (See Figure 1.4.)

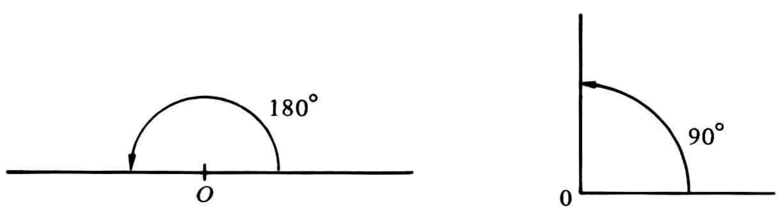


Figure 1.4 (a) Straight angle (b) Right Angle

An angle is **acute** if it is less in size than a right angle and is **obtuse** if it is larger than a right angle but smaller than a straight angle (see Figure 1.5). Figure 1.6 shows two angles that are larger than a straight angle.

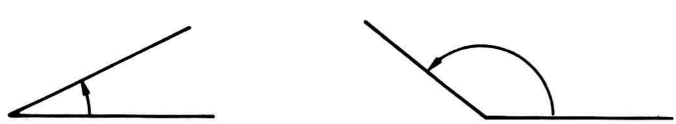


Figure 1.5 Acute Angle Obtuse Angle

Angles with the same initial and terminal sides are said to be **coterminal.** The two angles shown in Figure 1.7 are coterminal, but they are obviously not equal. Coterminal angles are sometimes considered equal, but there are many important