INTRODUCTORY MATHEMATICAL ANALYSIS

WITOLD A. J. KOSMALA

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INTRODUCTORY MATHEMATICAL ANALYSIS

To:
Vanessa, Konrad,
and my parents
Jerzy and Anna

Preface

This textbook is designed for a two-semester or a three-quarter course in the theory of calculus. The material can, however, be easily subdivided to suit any length course titled Introduction to Real Analysis, Theoretical Calculus, or the equivalent. This text is intended for college undergraduates, juniors or seniors, who have completed an entire calculus sequence, linear algebra, and preferably differential equations, and who wish to pursue careers in science, statistics, engineering, computer science, or business. Since branches in these areas have roots in real analysis, a solid background is essential. Giving students a solid background in theoretical undergraduate analysis is precisely the purpose of this text.

A large portion of the material covered should be somewhat familiar to students from their study of elementary calculus. Here, however, theory and deeper understanding is stressed. This text, without sacrificing depth and accuracy, is written in such a way that students at this level will succeed at and appreciate analytical mathematics. Through clear, accurate, and in-depth explanations, ideas are built upon, and the reader is adequately prepared for each new type of material. This text achieves a number of goals, making it most desirable in a classroom. First, students are introduced to proofs that motivate them toward clear thought and a total understanding of the material. Individual proofs include thorough explanations of what is being done and why and what goal is looked for—an essential item in beginning chapters. As the reader develops skill, the discussion of proofs gets shorter and less intuitive. Second, this text stimulates curiosity through the question, "Why?" Questioning encourages the reader to think about what has been said and either promotes logic or requires the reader to look back at previously covered material. Third, this text provides rigorous training in mathematical thinking through the omission of details at appointed places. The student is forced to read with pencil and paper. Fourth, this text excels in pointing out goals, outlining procedures, and giving good intuitive reasons; it is very easy to read. Also, detailed and complete graphs are used when necessary, as well as cross-referencing throughout the text. In addition, a number of sections that are usually not found in textbooks of this type are included, enhancing the text. The omission of optional sections, which are indicated by an asterisk, does not hinder the reader in understanding sections that follow. Historical notes also add a different perspective. And finally, expressions like "it can be shown," "obviously," "clearly," and so on, are used not to intimidate the reader but to simply point out that what follows should be easily grasped by the person reading the text. If not, then review of the necessary material is in order.

Each section in this textbook is followed by a set of exercises of varied difficulty. Exercises range from routine to creative and innovative, mixing theory and applications. Exercises at the end of each section are arranged in the same order as the preceeding material. In addition, every chapter is followed by a set of review problems of true/false nature. Often knowing what is true or false is more important than being able to prove a theorem. True statements need to be proven. A counterexample is requested for false statements. And a change is requested for those false statements that become true with a little help. Such problems foster deeper understanding of the material and stimulate curiosity. Students will become more creative and will no longer take statements for granted; instead they will learn to interrelate ideas. Each review section is followed by optional projects designed to improve students' creativity as well as reinforce ideas covered in previous chapters. These projects either continue the study of real analysis or bring out analysis in other areas of mathematics.

For quick reference, all definitions, examples, theorems, corollaries, remarks, and so on, are numbered in succession by chapter number, section number, and position in the sequence within each section. Hints and Solutions to Selected Exercises are located at the back of the text. But since most exercises may be completed in a variety of ways, I recommend viewing this part of the text only after exercises are attempted. A convenient index of symbols is also included in the back of the book. A supplement to this text is the *Instructor's Solutions Manual.*

Introductory Mathematical Analysis is an exceptional textbook with lots of flexibilities and is appropriate for any type of a college/university, most preferably competitive mathematics programs. Honor classes and classes with varied levels of ability will enjoy this text. I hope by using this textbook in the classroom, the failure level in the introductory analysis classes will drop and more students will be attracted to mathematics. Perhaps this textbook will help to increase the number of majors in areas that involve mathematics.

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Dr. Witold A. J. Kosmala

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1

Introduction

1.1 *Algebra of Sets*

Chapter 1 serves as an introduction to this textbook; terminology for the chapters that follow as well as methods of proving mathematical statements are discussed. Although the intent is to cover only the topics that are needed in the following chapters, we must realize that any material covered in this text can always be extended indefinitely.

Almost every math textbook begins with a study of sets, since the idea is fundamental to any area of mathematics. By a *set* we mean a collection of well-defined objects called *elements* or *members* of the set. And by *well defined* we mean that there is a definite way of determining whether or not a given element belongs to the set. To write a set it is customary to use braces { } with elements of the set listed inside them. Lowercase letters are usually used to represent the elements, whereas capital letters denote sets themselves. If an element r belongs to the set A, then we write $r \in A$. And if r is not an element of the set A, then we write $r \notin A$. Thus, if $A = \{1, 2, 3\}$, then $1 \in A$, but $4 \notin A$.

At times it is difficult to list all members of a set. For example, if the set B consists of all of the counting numbers smaller than 100, we usually write

$$B = \{1, 2, 3, \ldots, 99\} \text{ or}$$
$$B = \{x \mid x \text{ is a counting number smaller than } 100\}$$

The second of these is read as: B is the set of all elements x, such that x is a counting number smaller than 100. Thus, we described what is in the set instead of listing the elements. When using the descriptive method, we have to be sure that what we want to include in the set is indeed what we describe and is regarded by everyone in the same way. For example, the set $C = \{\text{all young men in the USA}\}$ is not a well-defined set since the word "young" has different meanings to different people.

There are many ways of describing any one particular set. For example, the set $D = \{2, 3\}$ can also be written as

$$D = \{x \mid x^2 - 5x + 6 = 0\},$$
$$D = \{x \mid x \text{ is a prime number less than } 4\}, \text{ or}$$
$$D = \{x \mid x \text{ and } x + 1, \text{ or } x \text{ and } x - 1 \text{ are prime numbers}\}$$

Sets used throughout this text are

$$N = \text{natural (counting) numbers} = \{1, 2, 3, \ldots\},$$
$$W = \text{whole numbers} = \{0, 1, 2, \ldots\},$$
$$Z = \text{integers} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$
$$Z_k = \{x \mid x \in Z \text{ and } x \geq k \in Z\} = \{x \in Z \mid x \geq k \in Z\}, \text{ and}$$
$$Q = \text{rational numbers} = \{x \mid x = \tfrac{p}{q} \text{ where } p, q \in Z \text{ and } q \neq 0\},$$

where $p, q \in Z$ means that $p \in Z$ and $q \in Z$,

$$Q^+ = \{x \in Q \mid x > 0\},$$
$$\Re = \Re^1 = \text{real numbers},$$
$$\Re^+ = \{x \in \Re \mid x > 0\}, \text{ and}$$
$$\Re^- = \{x \in \Re \mid x < 0\} = \{-x \mid x \in \Re^+\}$$

Not specifying where an element, say x, is automatically implies $x \in \Re$. Real numbers that are not rational are called *irrational.* In fact, there are more irrational than rational numbers. Elements in $\Re^+$ and elements in $\Re^-$ are called *positive* and *negative* real numbers, respectively. Elements in $\Re^+ \cup \{0\}$ and $\Re^- \cup \{0\}$ are called *nonnegative* and *nonpositive* real numbers, respectively. *Prime* numbers are natural numbers divisible by exactly two natural numbers, 1 and the natural number itself. An integer, p, is *even* if p can be written as $p = 2s$ for some integer, s. And an integer, q, is *odd* if it can be written as $q = 2t + 1$ for some integer, t. Sections 1.6 and 1.7 discuss the real numbers in greater depth.

The following are sets denoting *intervals,* where a and b are real numbers with $a < b$.

$$[a,b] = \{x \mid a \leq x \leq b\} \qquad (a,b) = \{x \mid a < x < b\}$$
$$[a,b) = \{x \mid a \leq x < b\} \qquad (a,b] = \{x \mid a < x \leq b\}$$
$$[a,\infty) = \{x \mid x \geq a\} \qquad (a,\infty) = \{x \mid x > a\}$$
$$(-\infty,a] = \{x \mid x \leq a\} \qquad (-\infty,a) = \{x \mid x < a\}$$

The symbols ∞ and $-\infty$ are called *infinity* and *minus infinity,* respectively. These symbols do not represent real numbers. Often, ∞ is written as $+\infty$. Note that we can write $\Re = (-\infty,\infty)$ and $\Re^+ = (0,\infty)$.

Definition 1.1.1

If A and B are sets, then A *equals* B—i.e., $A = B$—if and only if both sets consist of exactly the same elements.

The expression *if and only if,* commonly written by instructors as "iff," means that two statements are equivalent. This expression always applies to definitions, although it is often not written. Further, $A = B$ if and only if whenever $x \in A$, then $x \in B$, and whenever $x \in B$, then $x \in A$. If $A \neq B$, then the sets A and B are distinct—i.e., some element exists, say x, such that $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$.

Definition 1.1.2

If A and B are sets such that every element of A is also an element of B, then A is a *subset* of B, meaning A is contained in B and is denoted by $A \subseteq B$. If A is a subset of B but $A \neq B$, then A is called a *proper subset* of B and is denoted by $A \subset B$.

Thus, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. If a set has no elements in it, then it is called an *empty* or *null set* and is denoted by ϕ. The vacuous implication of Definition 1.1.2 is that the empty set is a subset of every set. We are now ready to make new sets from given ones.

Definition 1.1.3

If A and B are sets, then

(a) A *intersection* B, denoted by $A \cap B$, is the set of all elements that belong to both A and B. That is, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

(b) A *union* B, denoted by $A \cup B$, is the set of all elements that belong to either A or B. That is, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

(c) the *complement* of B in A, also referred to as the complement of B relative to A, or A *minus* B, denoted $A \setminus B$, is the set of all elements in A that are not in B. That is, $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

(d) A and B are *disjoint* if they have no elements in common. That is, $A \cap B = \phi$.

The shaded regions in Figure 1.1.1 represent Definition 1.1.3, parts (a)–(c).

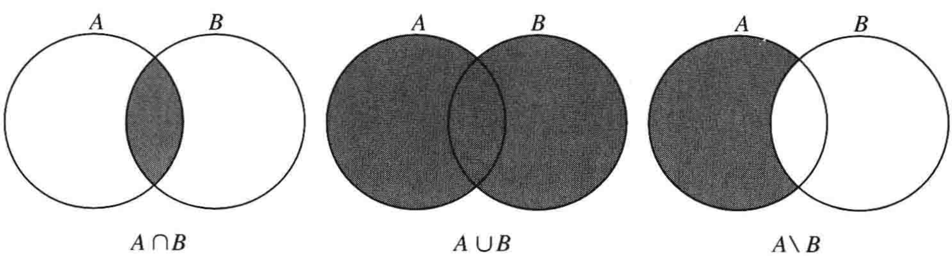

Figure 1.1.1

Example 1.1.4

If $A = \{1, 3, 5\}$ and $B = \{1, 3, 8\}$, find the following:

(a) $A \cap B$ **(b)** $A \cup B$ **(c)** $A \setminus B$

Answer $A \cap B = \{1, 3\}$, $A \cup B = \{1, 3, 5, 8\}$, and $A \setminus B = \{5\}$. Elements are never written more than once in a set. ■

The symbol ■ is used to indicate that a proof, answer, or remark is complete.

THEOREM 1.1.5 *If A, B, and C are sets, then*

(a) $A \cap A = A$ *and* $A \cup A = A$ *(Idempotent property)*

(b) $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$ *(Commutative property)*

(c) $(A \cap B) \cap C = A \cap (B \cap C)$ *and* $(A \cup B) \cup C = A \cup (B \cup C)$ *(Associative property)*

(d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(Distributive property)*

(e) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ *and* $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ *(De Morgan's[1] laws)*

Proof of first equality in part (d) Since we are to prove that two sets are equal, we need to show that one set is a subset of the other and vice versa. To prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, pick an arbitrary element, say $x \in A \cap (B \cup C)$. We will show that $x \in (A \cap B) \cup (A \cap C)$. By Definition 1.1.3, part (a), $x \in A$ and $x \in B \cup C$. Thus, $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. Hence, $x \in A \cap B$ or $x \in A \cap C$. And by Definition 1.1.3, part (b), $x \in (A \cap B) \cup (A \cap C)$.

Now, we need to prove conversely that if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap (B \cup C)$. This x is not necessarily the same as the one above. Since $x \in (A \cap B) \cup (A \cap C)$, $x \in A \cap B$ or $x \in A \cap C$. Thus, $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. In other words, $x \in A$, and either $x \in B$ or $x \in C$. Therefore, $x \in A$ and $x \in B \cup C$. Hence, $x \in A \cap (B \cup C)$. And the proof of the first equality in part (d) is complete.

Proof of first equality in part (e) Again, we need to show that one set is a subset of the other, and vice versa, to give equality of the two sets. Let $x \in A \setminus (B \cup C)$. We need to show that $x \in (A \setminus B) \cap (A \setminus C)$. Now since $x \in A \setminus (B \cup C)$, by Definition 1.1.3, part (c), $x \in A$ and $x \notin B \cup C$. Hence, $x \in A$, $x \notin B$, and $x \notin C$. Thus, $x \in A \setminus B$ and $x \in A \setminus C$. Therefore, $x \in (A \setminus B) \cap (A \setminus C)$, and so $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

To prove inclusion in the other direction, pick an arbitrary element, $x \in (A \setminus B) \cap (A \setminus C)$. Then, $x \in A \setminus B$ and $x \in A \setminus C$. Therefore, $x \in A$, $x \notin B$, and $x \notin C$. And so $x \in A$ and $x \notin B \cup C$. Hence, $x \in A \setminus (B \cup C)$. Thus,

[1]Augustus De Morgan (1806–1871), an English mathematician and logician born in India, is recognized for the development of the foundations of algebra, arithmetic, probability theory, and calculus.

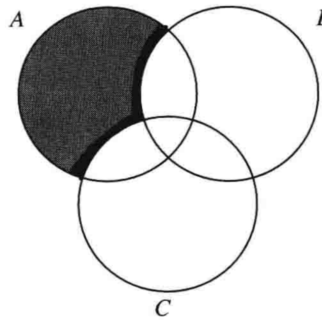

Figure 1.1.2

$(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$. And the proof of the first equality in part (e) is complete. See shaded region in Figure 1.1.2. ■

In view of Theorem 1.1.5, part (c), we can write $A \cap (B \cap C) = A \cap B \cap C$ and $A \cup (B \cup C) = A \cup B \cup C$. Furthermore, if $A_1, A_2, \ldots, A_n$ are n sets, where n is some fixed natural number, then

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{k=1}^{n} A_k = \{x \mid x \in A_k \text{ for all } k = 1, 2, \ldots, n\} \text{ and}$$

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{k=1}^{n} A_k = \{x \mid x \in A_k \text{ for some } k = 1, 2, \ldots, n\}$$

Definition 1.1.6

If A and B are two nonempty sets, then the *Cartesian product* (*cross product*) $A \times B$ is the set of all *ordered pairs* (a,b) such that $a \in A$ and $b \in B$. That is,

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Remark 1.1.7

The Cartesian product leads to the *Cartesian* (*rectangular*) *coordinate system,* named after Descartes,[2] where a point, P, in the plane $\Re^2 = \{(x,y) \mid x, y \in \Re\}$ is called an ordered pair (a,b). To locate a point (a,b) we begin by drawing horizontal and vertical number lines called the x-axis and y-axis, respectively. Then a units on the x-axis and b units on the y-axis are measured off to form a rectangle. This is how the name rectangular coordinate system came about. If one of a or b equals zero, then the rectangle becomes a line segment, and if both a and b equal zero, then the rectangle degenerates to a point called the

---

[2]René Descartes (1596–1650), a French mathematician and philosopher, is known in Latin as Cartesius. Descartes founded analytic geometry, introduced the symbols of equality and square and cube roots, proved how roots of an equation are related to sign changes within it, and did some work in physics.

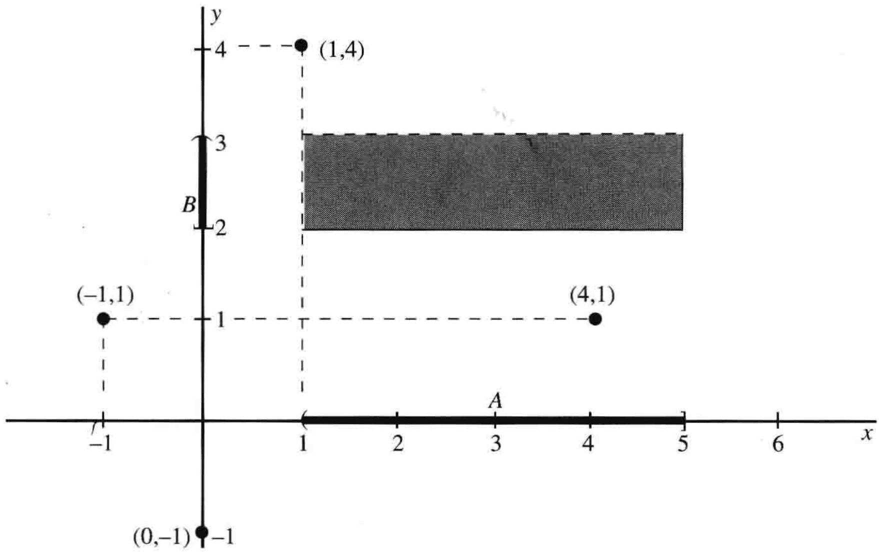

Figure 1.1.3

origin. The value a is called the *x-coordinate* (*first coordinate*), whereas the value b is called the *y-coordinate* (*second coordinate*). Often, the value a alone is referred to as a point. The same is true of b. Thus, whenever referring to a *point,* we either mean some real number or an ordered pair.

In Figure 1.1.3 the shaded region represents the Cartesian product $A \times B$ of two sets A and B, where A is the interval $(1,5]$ and B the interval $[2,3)$. A dotted line indicates that the points on the line are not included. Ordered pairs $(4,1)$, $(1,4)$, $(0,-1)$, and $(-1,1)$ are indicated by dots in the plane. In general, an ordered pair (b,a) is not the equivalent of ordered pair (a,b). Two ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$. Thus, $(a,b) = (b,a)$ if and only if $a = b$ and $b = a$. Hence, the point represented by this ordered pair must lie on a line passing through the origin and rising at 45°. ■

Exercises 1.1

1. Suppose that $A = \{1, 3, 5\}$, $B = \{1, 2, 4\}$, and $C = \{1, 8\}$. Find the following:

(a) $(A \cup B) \cap C$ **(b)** $A \cup (B \cap C)$

(c) $(A \setminus C) \cup B$ **(d)** $(A \cap B) \times C$

(e) $C \times C$ **(f)** $\{\phi\} \cap A$

2. Find real numbers x and y so that the ordered pair $(x - 2y, -2x + 2y) = (-4,2)$.

3. Prove Theorem 1.1.5.