COMMENTATIONES MATHEMATICAE

TOMUS SPECIALIS IN HONOREM

LADISLAI ORLICZ

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I

The present volume of Commentationes Mathematicae is dedicated to its Editor, Professor Władysław Orlicz, on the occasion of his 75-th birthday.

Professor Władysław Orlicz, an outstanding mathematician, was as a member of the famous Lwów School among the founders of modern Functional Analysis. The articles collected in this volume, many of them by Professor Orlicz's former students, were inspired by the ideas raised in the Work of Professor Orlicz. It is, we believe, the best symbol of respect that his friends world wide hold for Professor Orlicz.

Editorial Committee

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WANDA MATUSZEWSKA (Poznań)

Władysław Orlicz
A review of his scientific work

Władysław Orlicz was born on 24th of May 1903 in a small town, Okocim, lying in the district of Cracow. His parents, Franciszek and Maria née Rossknecht, had five sons; his father worked in the administration of the Okocim Brewery, and died when Władysław was only a few years old. The youngest of the brothers, Zbigniew, was killed during the first World War, the eldest, Kazimierz, became a victim of the second. The remaining three brothers, Tadeusz, Władysław and Michał, chose scientific careers. The eldest, Tadeusz, devoted himself to engineering, Władysław to mathematics, and the youngest, Michał, to geography. In 1919, the Orlicz family moved to Lvov, where Władysław Orlicz completed his mathematical studies at the Jan Kazimierz University. In the years 1920–1939, Stefan Banach and Hugo Steinhaus created in Lvov a mathematical centre of international importance, and Władysław Orlicz was fortunate enough to find himself in the company of many prominent mathematicians, who formed the famous School of Stefan Banach. At this time he had among his colleagues such mathematicians as Zygmunt Wilhelm Birnbaum, Marek Kac, Stanisław Mazur, Juliusz Schauder and Stanisław Ulam. In 1928, Władysław Orlicz obtained the degree of Doctor of Philosophy on the grounds of a thesis in the theory of orthogonal series; in 1933, he obtained the so-called *veniam legendi* (habilitation). From 1922 until 1937, he held posts as a scientific worker, first at the Institute of Mathematics of the Jan Kazimierz University, and then at the Technical University of Lvov.

In 1937 Władysław Orlicz was nominated associate professor of the University of Poznań, and the beginning of the second World War saw him at this post. He spent the years of the war in Lvov, where for some time he performed the duties of a professor at the Soviet State University. During the occupation of Lvov by the Germans, he took part in Polish underground education at grammar-school and university levels. In May of 1945 he returned to Poznań, since then has been associated with mathematics in Poznań. In 1948, Władysław Orlicz obtained the degree of full professor. In 1956 he became a corresponding member and in 1961 a full member of the Polish Academy of Sciences. During over thirty years of scientific and educational activity at the Adam Mickiewicz University, at the Mathematical Institute of the Polish Academy of Sciences, and in the Polish Mathematical Society, Władysław Orlicz contributed in a substantial manner to the development of mathematics in Poznań. Under his personal supervision, several dozen of mathematicians completed their doctoral dissertations. Over a dozen of his closest pupils obtained the title of professor or dozent. Such well-known professors as Andrzej Alexiewicz, Zbigniew Ciesielski, Julian Musielak, Jerzy Seidler, Zbigniew Semadeni, Roman Taberski, and others, are among his many pupils. It is also worth mentioning that Władysław Orlicz is editor of two periodicals, Studia Mathematica and Commentationes Mathematicae. In appreciation of his scientific merit, Władysław Orlicz has been given high state distinctions. He has also obtained many awards, among them state awards of the first and second degree, the award of the city and district of Poznań, and the award of the Alfred Jurzykowski Foundation. He is a doctor honoris causa of York University in Canada.

I will now present, in a concise manner, the scientific achievements of Władysław Orlicz, contained in his 136 scientific papers, the first of which appeared in 1926. Above all I would like to discuss those of his results which can now be found in specialist monographs of various authors, and those which have particularly attracted the attention of mathematicians, inspiring further research. I will thus mention the following trends of research.

General theory of orthonormal series. Real functions. W. Orlicz devoted to the general theory of orthogonal systems a number of papers written in the years between the two World Wars, and was one of the first mathematicians to apply methods of functional analysis in this field. He was primarily interested in the convergence of orthogonal series and in structural properties of sets of Fourier coefficients of functions of various classes.

Let (φ_n) be an arbitrary orthonormal system in (a, b). According to the classical theorem of Rademacher–Menshov, convergence of the series $\sum a_n^2 (\lg n)^2$ implies convergence almost everywhere of the series

$$(*) \qquad \sum_{n=1}^{\infty} a_n \varphi_n(x).$$

In [3] W. Orlicz has shown that supposing the series $\sum a_n^2 (\lg n)^2 w(n)$ to be convergent, where the sequence $w(n)$ increases to ∞ and is such that $\sum w(n_k)^{-1} < \infty$, $\sup \lg n_{k+1}/\lg n_k < \infty$ for an increasing sequence of indices

n_k (e.g. $w(n) = (\lg\lg n)^{1+\varepsilon}$), series (*) is unconditionally convergent almost everywhere. This means its convergence almost everywhere for each permutation of terms.

Some general remarks on unconditional convergence are given in [2]. The following problem has been considered: is the Orlicz criterion also a necessary condition of unconditional convergence almost everywhere of series (*) for arbitrary orthonormal systems? Positive results in this direction were obtained by Hungarian mathematicians (Leindler, Tandori) as late as 35 years after the paper of W. Orlicz.

Let S be a class of functions in (a, b) and let $\Omega(S)$ be the set of sequences of Fourier coefficients (a_n) of functions from S with respect to an orthonormal system (φ_n). In the theory of orthonormal series it is of interest to ask about the structure of various $\Omega(S)$; however, there are only a few results so simple and satisfactory as in the case of $S = L^2$. In [14] W. Orlicz proved that if $\sum p_n^2 = \infty$, then there exists a continuous function such that $\sum |a_n|\,|p_n| = \infty$, under the assumption that the system (φ_n) is infinite and consists of uniformly bounded functions. Hence follows in particular the existence of a continuous function for which the series $\sum |a_n|^{2-\varepsilon}$ is divergent for every $\varepsilon > 0$ (an analogous singularity in the case of the trigonometric system was shown first by Carleman). Another result concerning investigation of the structure of $\Omega(S)$ is the following theorem [33]: *if the system (φ_n) consists of continuous functions and the linear combinations of φ_n form a dense set in $C(a, b)$, and if $\sum p_n^2 = \infty$, then, for almost every distribution of signs $\pm$, the series $\sum \pm p_n \varphi_n(x)$ is not an expansion of a function from $L^1_{\langle a,b\rangle}$*. In connection with investigations of structures of various $\Omega(S)$, one considers the so-called multiplicators which map a set $\Omega(S_1)$ in $\Omega(S_2)$. Multiplicators for a general orthogonal system (and also the so-called majorants) were investigated by W. Orlicz in papers [3], [4], [14], [21], [33], [34]. Among the theorems of another type concerning arbitrary complete systems (φ_n), the following one is worth mentioning [3]: if the system (φ_n) is complete in $L^2_{\langle a,b\rangle}$, then the series $\sum \varphi_n^2(x)$ is divergent almost everywhere. This theorem is useful for example in constructing orthogonal series which show various divergence phenomena.

Three papers [41], [37], [52], prepared during the second World War but published in the first years after the war, are devoted to the problem of existence of continuous non-differentiable functions. In [37] W. Orlicz considers continuous functions of the form $f_\varepsilon(x) = \sum_1^\infty \varepsilon_n f_n(x)$, where $\varepsilon = \{\varepsilon_n\}$ are sequences with terms $+1$, -1 and the series $\sum_1^\infty |f_n(x)|$ is uniformly convergent in $\langle a, b\rangle$. Under suitable assumptions on $f_n(x)$ it is proved that the function $f_\varepsilon(x)$ is non-differentiable in $\langle a, b\rangle$, in a set of sequences ε residual in a suitably chosen metric space of sequences. Another theorem states that, for almost all ε, the function $f_\varepsilon(x)$ has no derivative almost everywhere. Let ω, ω_1 be non-decreasing functions for $0 \leqslant h \leqslant 1$, equal to zero exactly at $h = 0$ and tending to zero as $h \to 0$. The following problem is considered in [41]: under what assumptions on ω, ω_1 there exists a function f of period 1 such that

$$(*) \qquad |f(x+h)-f(x)| \leqslant \omega(h),$$

$$(**) \qquad \limsup_{h\to 0} |f(x+h)-f(x)|/\omega_1(h) = \infty,$$

simultaneously for every x. It turns out that a necessary and sufficient condition for the existence of a function with the above mentioned properties is

$$\liminf_{h\to 0} \frac{\omega_1(h)}{h}\gamma(h) = 0, \qquad \text{where} \qquad \gamma(h) = \sup_{0<k\leqslant h} \frac{k}{\omega(k)}.$$

The methods applied in [41] are mostly constructive. Analogous problems, but replacing lim sup by lim supas in (**), are considered in [52]. In the above mentioned papers devoted to non-differentiable functions, W. Orlicz considered functional series whose terms are provided with random signs $+1$, -1 or 1, 0. He turned to some investigations connected with series of this type in [50], with the purpose of applying such series in some problems of divergence of orthogonal series or problems of non-differentiability of functions. Let us quote one of the results given in this paper.

Let f_n^i be measurable functions in (a, b). Let us suppose that (a) the series $F_i(x) = \sum [f_n^i(x)]^2$, $i = 1, 2, \ldots$, are convergent almost everywhere in (a, b), (b) $f_n^i(x)$ tend to $f_n(x)$ in measure in (a, b). Let us form the series

$$(+) \qquad F_i(x, \varepsilon) = \sum_1^\infty \varepsilon_n f_n^i(x)$$

for $i = 1, 2, \ldots$, where $\varepsilon = \{\varepsilon_n\}$, $\varepsilon_n = +1, -1$. It is known that under assumption (a) the series $F_i(x, \varepsilon)$ are convergent almost everywhere for almost every ε. The following theorem holds: *There exists a measurable set C in (a, b) such that almost every distribution of signs ε has the following property: the sequence of series (+) is convergent in measure on C, and if $m(C) < b-a$, then for every subset of positive measure of $(a, b)\setminus C$ the sequence of series (+) is divergent in measure.* If M denotes the maximal set of convergence in measure of the sequence of series $\sum_1^\infty |f_n^i(x) - f_n(x)|^2$, then neglecting a set of measure zero we have $C \subset M$.

General theory of linear methods of summability. The first applications of methods of functional analysis to the theory of summability are due to S. Banach and S. Mazur. However, an essential step forward was made jointly by S. Mazur and W. Orlicz in 1932. Their results were published in part, without proofs, in a note in the C. R. Acad. Sci. Paris [19], but full publication [58] was delayed until 1954 for various reasons. In the meantime, problems similar to those considered in [19] and [58] were investigated by various mathematicians (Agnew, Brudno, Darewsky, Wilansky, Zeller and others). Especially in papers of K. Zeller both the subject of investigations and the methods of functional analysis applied are close to those of [58]. The most important observation of Mazur and Orlicz in 1932 was that Banach spaces are not a satisfactory tool in investigations of fields of matrix methods of summability, and that fields of summability belong to a more general type of spaces, called *spaces* B_0 by Mazur and Orlicz at that time (today such spaces are also called *countably normed spaces*). Spaces B_0 were later investigated by S. Mazur and W. Orlicz in other papers ([46], [53]). Paper [58] is devoted, first of all, to the following problems: consistency of summability methods, orders of growth of sequences from a field of summability, existence of unbounded sequences in a field of summability. The most famous theorem is the Mazur–Orlicz theorem on the consistency of matrix methods of summability for bounded sequences: *if every bounded sequence summable by a permanent method A is summable by a permanent method B, then the methods A and B both sum bounded sequences to the same limit.* Similar consistency theorems for double sequences are to be found in papers by A. Alexiewicz and W. Orlicz [62], [80], and for bounded sequences with values in a Banach space in paper [79]. The above mentioned papers prepared jointly with Alexiewicz were written after the second World War, when the interests of W. Orlicz rather were shifting in another direction. However, in connection with the theory of modular spaces and the theory of Saks spaces, W. Orlicz considered also problems connected with the theory of summability (e.g. in papers [68], [73], [99], [101]).

Metric locally convex vector spaces. Investigations of fields of summability methods led S. Mazur and W. Orlicz in the years 1932–1936 to establishing the foundations of the theory of metric locally convex vector spaces, which they called B_0-*spaces*. They only published these results, in papers [46], [53], after the second World War, and by that time a large part of them had been discovered independently by French mathematicians. An important and best known result of paper [53] is the following Mazur–Orlicz theorem, which is a generalization of the Hahn–Banach theorem on the extension of linear functionals: Let ω

be a subadditive, positive-homogeneous functional defined in a vector space X, and let (x_t), $t \in T$, be a family of elements of the space X, and (c_t), $t \in T$ — a family of real numbers. The necessary and sufficient condition for the existence of an additive, homogeneous functional ξ on X satisfying conditions

$$\xi(x_t) \geqslant c_t \quad \text{for } t \in T, \qquad \xi(x) \leqslant \omega(x) \quad \text{for } x \in X,$$

simultaneously, is the following one: for every finite system of indices $t_1, \ldots, t_n \in T$ and for arbitrary numbers $\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0$, the following inequality holds:

$$\sum_{i=1}^{n} \lambda_i c_{t_i} \leqslant \omega\Big(\sum_{i=1}^{n} \lambda_i x_{t_i}\Big).$$

Of the other results contained in [53], we should mention the problems concerning the solution of inequalities with linear operators.

Polynomial operators. Papers [24], [25], [30] of S. Mazur and W. Orlicz, concerning polynomial operators, were of pioneer character. In [24], the considerations centred around three equivalent definitions of homogeneous polynomial operators by means of which one defines arbitrary polynomial operators as finite sums of homogeneous polynomial operators. The main result of [25] is that of transferring the Banach–Steinhaus theorem on sequences of linear operators to the case of polynomial operators with uniformly bounded degrees. The authors do not limit themselves to the case where the inverse images of the operators are Banach spaces, but they assume them to be F-spaces. Even in the case of linear operators this theorem is an interesting generalization of the classical Banach–Steinhaus theorem, because it requires an applications of the notion of bounded sets in F-spaces, which first appeared in [20] in 1933. In papers [30], [31] S. Mazur and W. Orlicz give a number of theorems on divisibility of operators.

Saks spaces. From 1950 on, W. Orlicz was interested in Saks spaces, which he defines in the following manner. Let two norms $\|\ \|$, $\|\ \|^*$ be defined in a vector space X, the first one being homogeneous and the second one being generally an F-norm. The unit ball $\|x\| \leqslant 1$ metrized by means of the formula for distance $d(x', x'') = \|x' - x''\|^*$ was called by W. Orlicz a *Saks set*, and if it is complete — a *Saks space*. Saks sets and Saks spaces are closely connected with two-norm spaces, but none of these notions is contained in any of the others. The main advantage of Saks spaces is the possibility of application of the Baire theorem. For Saks spaces W. Orlicz investigated mostly continuous maps and sequences of linear operators on Saks spaces. There are interesting analogies of the classical Banach–Steinhaus theorem in Saks spaces, because they are useful in the theory of orthogonal expansions and in the theory of summability; for example, they may be applied in order to prove consistency theorems for various summability methods for bounded sequences ([43], [47], [54], [64], [66]–[69], [71], [73], [92], [132]).

Unconditional convergence in normed vector spaces. In paper [5] in 1929 W. Orlicz introduced the important notion of unconditional convergence of a series of elements. Let X be a normed vector space or, more generally, a linear topological space. A series $\sum x_n$ is called *unconditionally convergent* if it is convergent for each arrangement of its terms. Unconditional convergence is equivalent to convergence of each partial series (*) $\sum x_{n_k}$. W. Orlicz in his later papers called this last property *perfect convergence* of the series of elements x_n (now called also *subseries convergence*). The series $\sum x_n$ is called *perfectly bounded* if the set of sums $x_{n_1}+x_{n_2}+\ldots+x_{n_k}$ is bounded, where $n_1, n_2, \ldots, n_k$ are different indices. The series $\sum x_n$ has the *property* 0 if every perfectly bounded series is perfectly convergent. W. Orlicz has shown in [5] that every sequentially weakly complete Banach space possesses the property 0. The method of proof applied in [5] leads, after a non-essential modification, to the following important result: let X be a Banach space; then a series $\sum x_n$ is perfectly convergent if and only if every subseries $\sum x_{n_k}$ is weakly convergent to an element. This theorem was known from W. Orlicz's communications in Lvov, in the period between the two World Wars, and in this form it was given without proof in the appendix to a monograph of S. Banach, *Théorie des opérations linéaires* (1933). Since the proof of this beautiful theorem was first published by Pettis in 1938, it is often quoted as the Orlicz–Pettis theorem. Other deep theorems of W. Orlicz concern perfect boundedness of series in some function spaces [16]–[18], [38], [63], [72]. For example, W. Orlicz proved that in spaces L^p, perfect boundedness of the series $\sum x_n$ implies convergence of $\sum \|x_n\|^p$, if $p \geqslant 2$, convergence of $\sum \|x_n\|^2$, if $1 \leqslant p \leqslant 2$. In the space of measurable functions, perfect boundedness of a series of elements (in the sense of the respective F-norm) implies convergence of the series $\sum x_n^2(t)$ almost everywhere. In a paper written jointly with W. Matuszewska [115] there were investigated, for normed function spaces necessary and sufficient conditions for a space to have the property 0. It turns out that a necessary condition is here the absolute continuity of the norm, and this condition is sufficient for a class of modular spaces. W. Orlicz's ideas connected with the notions of perfect convergence and of the property 0 inspired the work of many mathematicians and are of great importance in the theory of vector-measure and the integral.

Vector-valued functions, Baire category method, theory of measure and integral. The Baire category method was applied by W. Orlicz in proofs of the existence of continuous functions possessing different kinds of singularities ([37], [41], [52]). In a paper written jointly with W. Matuszewska he applied this method in order to prove the non-efficiency of some elementary tests for the convergence of Fourier series [81]. Most striking is the application of this method to ordinary differential equations

$$y' = f(x, y). \qquad (*)$$

Considering continuous functions f in a rectangle Q, the set of functions f from the metric space $C(Q)$ for which equation $(*)$ possesses a unique solution in Q is residual in $C(Q)$ [15]. A similar result for partial differential equations of a hyperbolic type is given by A. Alexiewicz and W. Orlicz in [65].

In two papers vector-valued functions with values in Banach spaces are considered. The first one, a paper written jointly with A. Alexiewicz [95], is devoted to weak Baire classification, and the second one [81], written together with W. Matuszewska — to the Riesz–Fischer theorem for vector-valued functions. In the years 1967–1970, one of the subject of W. Orlicz's papers were finitely additive vector-valued integrals and measures [112]–[114], [119], [121], [124], [125]. In these papers W. Orlicz investigated absolute continuity and weak absolute continuity of additive set functions $x(:)$ defined e.g. on a ring $(\mathscr{C})$ of sets, with respect to a submeasure $\eta(:)$. The term "weak absolute continuity" means the following property: if $e_1 \supset e_2 \supset \ldots \supset e_n \ldots \in \mathscr{C}$, $\eta(e_n) \to 0$ as $n \to \infty$, then $x(e_n) \to 0$. At the same time W. Orlicz wrote also, jointly with L. Drewnowski, some papers concerning integral representations of some functional. The most important results are those of [123], where an integral representation of a functional ξ is given, defined on a normal vector lattice X of measurable functions under the assumption that ξ is orthogonally additive, i.e. $\xi(x+y) = \xi(x)+\xi(y)$ for $|x| \wedge |y| = 0$, $x, y \in X$, and the assumption of continuity in the sense of b_μ-convergence (a sequence $\{x_n\}$ is b_μ-convergent to x_0 if it is convergent to x_0 in measure μ and if $|x_n| \leqslant y$ for $n = 1, 2, \ldots$, where $y \in X$).

Orlicz spaces, modular spaces. At the very beginning of his research activity, W. Orlicz became interested in the class of normed spaces, now known under the name of Orlicz spaces. The cycle of papers concerning these problems was started by an extensive paper written in 1931 jointly with Z. W. Birnbaum [10], and the essential ideas were contained in papers [13] and [27]. Let φ be a convex function in $\langle 0, \infty)$ equal to zero exactly at zero. In [13] W. Orlicz assumes, moreover, that φ satisfies the so-called Δ_2 condition, the condition $\varphi(u)/u \to 0$ as $u \to 0$, and that

$\varphi(u)/u \to \infty$ as $u \to \infty$. These are typical conditions given by authors dealing with problems of this kind. By these assumptions on the function φ W. Orlicz introduces the class of functions x measurable in (a, b) and such that the integral $\int_a^b \varphi(|x(t)|)\,dt$ is finite. However, already in [27] W. Orlicz omits the condition Δ_2 and considers the vector space of functions for which

$$(*) \qquad \int_a^b \varphi(\lambda|x(t)|)\,dt < \infty \qquad \text{for } \lambda > 0 \text{ (dependent on } x).$$

He proves that this is a Banach space under a suitably chosen norm (called the *Orlicz norm* today). At present, this space is usually denoted by the symbol $L^{*\varphi}$. In the papers quoted above W. Orlicz investigated the following problems concerning spaces $L^{*\varphi}$: separability, representation of linear functionals, reflexivity, etc. After the second World War, W. Orlicz turned back to various problems connected with spaces $L^{*\varphi}$ in papers [60], [75], [89], [90], [94], [108]; the assumption of convexity is now replaced by a more general one: namely, that φ is a continuous, non-decreasing function in $\langle 0, \infty)$, equal to zero exactly at zero and tending to ∞ as $u \to \infty$ (such functions are called *φ-functions*). Now Orlicz spaces are defined as spaces $L^{*\varphi}$ whose elements are measurable functions satisfying condition $(*)$ (obviously, one may use a more general notion of integral) without the assumption of convexity of φ. In papers written jointly with W. Matuszewska [87], [107] there were introduced upper and lower indices σ_φ, s_φ describing the order of growth of the φ-function. They are useful in various problems. If we limit ourselves to convex φ-functions, then Orlicz spaces $L^{*\varphi}$ are examples of modular spaces in the sense of H. Nakano. Nevertheless, the theory of Orlicz spaces has its own specification, which differs from that of the theory of Nakano. Namely, the principal role is played by the φ-functions, which serve to generate the metrics and to formulate various theorems; moreover, in general no convexity assumption on φ is imposed. In the theory of Nakano, the fundamental role is played by the notion of a convex modular. W. Orlicz, together with a group of mathematicians in Poznań, developed the theory of modular spaces, starting with a more general notion of a modular than that used by the school of Nakano. In papers [78], [82], written jointly with J. Musielak, the notion of a modular was defined in the following manner: Let X be a real vector space. A functional ϱ on X assuming real values or ∞ is called a *modular* if the following conditions are satisfied: (a) $\varrho(x) = 0$ if and only if $x = 0$, (b) $\varrho(-x) = \varrho(x)$, (c) $\varrho(ax + \beta y) \leqslant \varrho(x) + \varrho(y)$ for $x, y \in X$, $a, \beta \geqslant 0$, $a + \beta = 1$. If X is a *vector lattice*, the above conditions are supplemented by the following one: $\varrho(x) \leqslant \varrho(y)$ for $|x| \leqslant |y|$. The set $X(\varrho) \subset X$ of elements $x \in X$ for which $\varrho(\lambda x) \to 0$