Chaos, Fractals, and Noise

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Stochastic Aspects of Dynamics

Second Edition

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To the memory of

Maria Ważewska-Czyżewska

Preface to the Second Edition

The first edition of this book was originally published in 1985 under the title "Probabilistic Properties of Deterministic Systems." In the intervening years, interest in so-called "chaotic" systems has continued unabated but with a more thoughtful and sober eye toward applications, as befits a maturing field. This interest in the serious usage of the concepts and techniques of nonlinear dynamics by applied scientists has probably been spurred more by the availability of inexpensive computers than by any other factor. Thus, computer experiments have been prominent, suggesting the wealth of phenomena that may be resident in nonlinear systems. In particular, they allow one to observe the interdependence between the deterministic and probabilistic properties of these systems such as the existence of invariant measures and densities, statistical stability and periodicity, the influence of stochastic perturbations, the formation of attractors, and many others. The aim of the book, and especially of this second edition, is to present recent theoretical methods which allow one to study these effects.

We have taken the opportunity in this second edition to not only correct the errors of the first edition, but also to add substantially new material in five sections and a new chapter. Thus, we have included the additional dynamic property of sweeping (Chapter 5) and included results useful in the study of semigroups generated by partial differential equations (Chapters 7 and 11) as well as adding a completely new Chapter 12 on the evolution of distributions. The material of this last chapter is closely related to the subject of iterated function systems and their attractors-fractals. In addition, we have added a set of exercises to increase the utility of the work for graduate courses and self-study.

In addition to those who helped with the first edition, we would like to thank K. Alligood (George Mason), P. Kamthan, J. Losson, I. Nechayeva, N. Provatas (McGill), and A. Longtin (Ottawa) for their comments.

A.L.
M.C.M.

Preface to the First Edition

This book is about densities. In the history of science, the concept of densities emerged only recently as attempts were made to provide unifying descriptions of phenomena that appeared to be statistical in nature. Thus, for example, the introduction of the Maxwellian velocity distribution rapidly led to a unification of dilute gas theory; quantum mechanics developed from attempts to justify Planck's ad hoc derivation of the equation for the density of blackbody radiation; and the field of human demography grew rapidly after the introduction of the Gompertzian age distribution.

From these and many other examples, as well as the formal development of probability and statistics, we have come to associate the appearance of densities with the description of large systems containing inherent elements of uncertainty. Viewed from this perspective one might find it surprising to pose the questions: "What is the smallest number of elements that a system must have, and how much uncertainty must exist, before a description in terms of densities becomes useful and/or necessary?" The answer is surprising, and runs counter to the intuition of many. A one-dimensional system containing only one object whose dynamics are completely deterministic (no uncertainty) can generate a density of states! This fact has only become apparent in the past half-century due to the pioneering work of E. Borel [1909], A. Rényi [1957], and S. Ulam and J. von Neumann. These results, however, are not generally known outside that small group of mathematicians working in ergodic theory.

The past few years have witnessed an explosive growth in interest in physical, biological, and economic systems that could be profitably studied using densities. Due to the general inaccessibility of the mathematical literature to the nonmathematician, there has been little diffusion of the concepts and techniques from ergodic theory into the study of these "chaotic" systems. This book attempts to bridge that gap.

Here we give a unified treatment of a variety of mathematical systems generating densities, ranging from one-dimensional discrete time transformations through continuous time systems described by integro-partial-differential equations. We have drawn examples from a variety of the sciences to illustrate the utility of the techniques we present. Although the range of these examples is not encyclopedic, we feel that the ideas presented here may prove useful in a number of the applied sciences.

This book was organized and written to be accessible to scientists with a knowledge of advanced calculus and differential equations. In various places, basic concepts from measure theory, ergodic theory, the geometry of manifolds, partial differential equations, probability theory and Markov processes, and stochastic integrals and differential equations are introduced. This material is presented only as needed, rather than as a discrete unit at the beginning of the book where we felt it would form an almost insurmountable hurdle to all but the most persistent. However, in spite of our presentation of all the necessary concepts, we have not attempted to offer a compendium of the existing mathematical literature.

The one mathematical technique that touches every area dealt with is the use of the lower-bound function (first introduced in Chapter 5) for proving the existence and uniqueness of densities evolving under the action of a variety of systems. This, we feel, offers some partial unification of results from different parts of applied ergodic theory.

The first time an important concept is presented, its name is given in bold type. The end of the proof of a theorem, corollary, or proposition is marked with a ■; the end of a remark or example is denoted by a □.

A number of organizations and individuals have materially contributed to the completion of this book.

In particular the National Academy of Sciences (U.S.A.), the Polish Academy of Sciences, the Natural Sciences and Engineering Research Council (Canada), and our home institutions, the Silesian University and McGill University, respectively, were especially helpful.

For their comments, suggestions, and friendly criticism at various stages of our writing, we thank J. Bélair (Montreal), U. an der Heiden (Bremen), and R. Rudnicki (Katowice). We are especially indebted to P. Bugiel (Krakow) who read the entire final manuscript, offering extensive mathematical and stylistic suggestions and improvements. S. James (McGill) has cheerfully, accurately, and tirelessly reduced several rough drafts to a final typescript.

Contents

1

Introduction

We begin by showing how densities may arise from the operation of a one-dimensional discrete time system and how the study of such systems can be facilitated by the use of densities.

If a given system operates on a density as an initial condition, rather than on a single point, then successive densities are given by a linear integral operator, known as the Frobenius–Perron operator. Our main objective in this chapter is to offer an intuitive interpretation of the Frobenius–Perron operator. We make no attempt to be mathematically precise in either our language or our arguments.

The precise definition of the Frobenius–Perron operator is left to Chapter 3, while the measure-theoretic background necessary for this definition is presented in Chapter 2.

1.1 A Simple System Generating a Density of States

One of the most studied systems capable of generating a density of states is that defined by the quadratic map

$$S(x) = \alpha x(1-x) \qquad \text{for } 0 \le x \le 1. \tag{1.1.1}$$

We assume that $\alpha = 4$ so S maps the closed unit interval $[0, 1]$ onto itself. This is also expressed by the saying that the **state** (or **phase**) **space** of the system is $[0, 1]$. The graph of this transformation is shown in Fig. 1.1.1a.

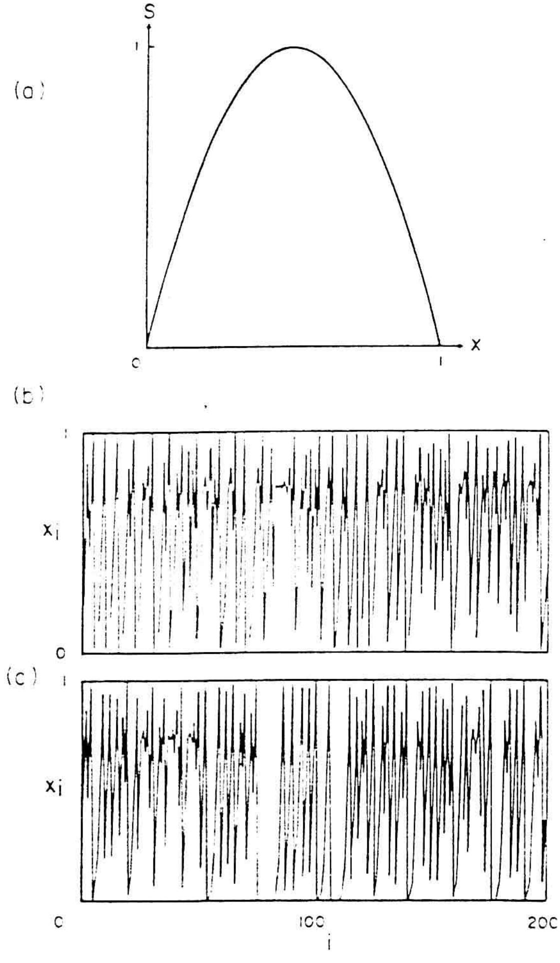

FIGURE 1.1.1. The quadratic transformation (1.1.1) with $\alpha = 4$ is shown in (a). In (b) we show the trajectory (1.1.2) determined by (1.1.1) with $x^0 = \pi/10$. Panel (c) illustrates the sensitive dependence of trajectories on initial conditions by using $x^0 = (\pi/10) + 0.001$. In (b) and (c), successive points on the trajectories have been connected by lines for clarity of presentation.

Having defined S we may pick an initial point $x^0 \in [0, 1]$ so that the successive states of our system at times $1, 2, \ldots$ are given by the trajectory

$$x^0, S(x^0), S^2(x^0) = S(S(x^0)), \ldots. \tag{1.1.2}$$

A typical trajectory corresponding to a given initial state is shown in Figure 1.1.1b. It is visibly erratic or chaotic, as is the case for almost all x^0. What is even worse is that the trajectory is significantly altered by a slight change

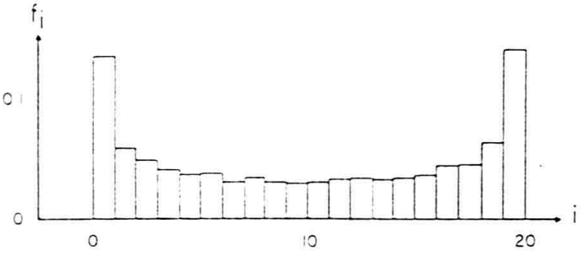

FIGURE 1.1.2. The histogram constructed according to equation (1.1.3) with $n = 20$, $N = 5000$, and $x^0 = \pi/10$.

in the initial state, as shown in Figure 1.1.1c for an initial state differing by 10^{-3} from that used to generate Figure 1.1.1b. Thus we are seemingly faced with a real problem in characterizing systems with behaviors like that of (1.1.1).

By taking a clue from other areas, we might construct a histogram to display the frequency with which states along a trajectory fall into given regions of the state space. This is done in the following way. Imagine that we divide the state space $[0, 1]$ into n discrete nonintersecting intervals so the ith interval is (we neglect the end point 1)

$$[(i-1)/n, i/n) \qquad i = 1, \ldots, n.$$

Next we pick an initial system state x^0 and calculate a long trajectory

$$x^0, S(x^0), S^2(x^0), \ldots, S^N(x^0)$$

of length N where $N >> n$. Then it is straightforward to determine the fraction, call it f_i, of the N system states that is in the ith interval form

$$f_i = \frac{n}{N}\{\text{number of } S^j(x^0) \in [(i-1)/n, i/n),\ j = 1, \ldots, N\}. \qquad (1.1.3)$$

We have carried out this procedure for the initial state used to generate the trajectory of Figure 1.1.1b by taking $n = 20$ and using a trajectory of length $N = 5000$. The result is shown in Figure 1.1.2. There is a surprising symmetry in the result, for the states are clearly most concentrated near 0 and 1 with a minimum at $\frac{1}{2}$. Repeating this process for other initial states leads, in general, to the same result. Thus, in spite of the sensitivity of trajectories to initial states, this is not *usually* reflected in the distribution of states within long trajectories.

However, for certain select initial states, different behaviors may occur. For some initial conditions the trajectory might arrive at one of the fixed points of equation (1.1.1), that is, a point x_* satisfying

$$x_* = S(x_*).$$