Introduction to College MATHEMATICS

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Holt, Rinehart and Winston, Inc.
New York Chicago San Francisco Atlanta
Dallas Montreal Toronto London

Library of Congress Catalog Card Number: 68-12012
2652600
Printed in the United States of America
1 2 3 4 5 6 7 8 9

Cover and text design by Rita Naughton

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Preface

This book is designed for a course in mathematics that assumes a background of intermediate algebra and plane geometry and progresses through introductory topics in the calculus of elementary functions of one variable. The subject matter is organized so as to provide a background for courses in linear algebra, abstract algebra, and finite mathematics as well as courses in analysis. The presentation attempts to bring more unity to the subject matter than is usually the case at this level of treatment. Hopefully, the student is given a feeling for the relatedness of mathematical ideas rather than simply an exposure to a miscellany of techniques.

Elementary logic is introduced informally with a minimum of symbolism and then used consistently in the subsequent developments. The study of the real number system involves some general notions of the structure of an abstract system. The structures of other systems such as matrices and subsets of a set are obtained, compared, and applied to the solutions of linear systems and discrete probability.

The elementary functions are approached as real mappings. Then the properties of trigonometric, exponential, and logarithmic functions evolve from the properties of the reals. Polynomials are treated both as algebraic forms and as functions. An introduction to Euclidean vector space allows a richer treatment of analytic geometry of lines and planes.

To motivate calculus, the problem of area is posed first. A development of sequences and series then leads to the definite integral. Thus a student is grounded in the definition of the definite integral before the introduction of the derivative and the antiderivative. Limits of functions are defined carefully, and then the problem of tangents is used to motivate the derivative. Procedures of the calculus are then applied to a variety of physical and geometric problems. In this context the conic sections and other aspects of analytic geometry are treated.

We wish to emphasize that the choice of topics and their ordering have been guided by the desire to establish a flow of mathematical ideas rather than a succession of isolated topics. A concept is presented only if it contributes to the general development of the subject matter or if it illustrates an idea that has been established.

As one example of this continuity of ideas, matrices arise out of the need for a systematic method of solving linear systems. Then the properties of operations on matrices provide an illustration of a noncommutative ring. Determinants are needed to answer the question of the existence of inverses of square matrices. Subsequently matrices are used in the study of relations and as representations of the linear mappings of the plane.

At an introductory level this text can be used with various categories of students. For a complete beginning course for pre-science students most of the topics could be covered in a one-year course. For other types of students the following choices are suggested.

A liberal arts course, for two semesters:
Chapters 1, 2, 3 (omitting Sections 3-1 and 3-6), 4, 5, 6, 7, 10, 11, 12, 13, 14

A course prerequisite to linear or abstract algebra, for one semester:
Chapters 1, 2, 3, 4, 5, 7, 11

A course in finite mathematics, for one semester:
Chapters 1, 2, 4, 5, 6, 7, 11

A course introductory to analysis, for one semester:
Chapters 1, 2, 3, 7, 8, 9, 10, 11

We acknowledge the helpful cooperation of our colleagues at Franklin and Marshall College in using portions of the book in mimeographed form in a variety of situations. Their comments and the reactions of the students were vital factors in the molding of the present version.

V.H.H. and D.W.W.

March 1968
Lancaster, Pennsylvania

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Contents

Chapter 1

Logic, Sets, and Relations

1-1 INTRODUCTION

This chapter is intended to help the reader organize his thinking into logical patterns and show him how to state exactly what he means. The logical principles and the terminology introduced for this will be used in the development of the following topics.

1-2 INFORMAL LOGIC OF SETS

The classifying of things into *sets* or classes is common. Some sets of people are: all persons enrolled for mathematics courses at accredited colleges; all citizens of the United States; all playing members of the San Francisco Giants; all persons of age 30 years or less; all persons over 10 feet tall. Some other sets are: all ink bottles in the White House; all ball-point pens that do not work; all possible rectangles in a plane; all even integers; all positive integers.

Some of the sets listed can be counted. During the regular baseball season the set of players in a National League team, such as the Giants, has 25 members. On the other hand, the set of all possible rectangles in a plane cannot be counted. The set of all people over 10 feet tall possibly has no members. A set having no members is called an *empty set* or a *null set* and denoted by the Greek letter ϕ (phi).

The word *element* designates an individual member of a set. When elements of one set may also be elements of another, various relationships exist between the sets. The set of all even integers is a *subset* of the set of all integers, since every even integer is also an integer. The set of all freshmen at a particular college is a subset of the total student body of that college. The set of all members of a mathematics class is not a subset of all freshmen if an upperclassman is in the class. In effect, one set is a subset of another, provided that every element of the one is also an element of the other. (More formal definitions will follow in Sections 1-9 and 5-1. For the present we shall rely on informal descriptions.)

Some concepts of deductive logic can be introduced informally with reference to relationships between sets. For instance, consider the following examples.

EXAMPLE 1

1. All *rectangles* are *parallelograms.*
2. All *parallelograms* are *polygons.*
3. All *rectangles* are *polygons.*

Each statement in Example 1 deals with two sets and makes an assertion about the relationship between their elements. Now, consider all three statements together and think about their implications. Even a person unacquainted with the meanings of the nouns in italics feels compelled to accept statement 3 as inescapable from the evidence given in statements 1 and 2. Statements 1 and 2 together, when used as evidence in this way, are called the *hypothesis.* Is statement 1 an inescapable conclusion from statements 2 and 3 taken as the hypothesis?

On the basis of plane geometry each statement in Example 1 is true. But the truth or falsity of statements is less important at this point than the pattern of reasoning which leads us to inescapable conclusions. This pattern becomes more evident when we substitute symbols or undefined terms for the nouns.

EXAMPLE 2

1. All x's are y's.
2. All y's are z's.
3. All x's are z's.

EXAMPLE 3

1. All marfets are trilos.
2. All trilos are scarpuls.
3. All marfets are scarpuls.

In these examples again, statement 3 is forced upon us as an inescapable conclusion if we grant the evidence given in statements 1 and 2. There can be no question of the truth or falsity of the individual statements, because in Example 2 only symbols of arbitrary classes are used, and in Example 3 unfamiliar and undefined names are used. Still a framework of reasoning remains that suggests the formulation of certain fundamental rules of logic. Every mathematical argument must start by agreeing on the use of undefined terms and rules of reasoning. Once these have been accepted, the process of arriving at inescapable conclusions from a stated hypothesis is called *deductive reasoning.*

Example 4 illustrates that, even with a simple hypothesis, it is not always easy to decide what inescapable conclusions, if any, follow.

EXAMPLE 4

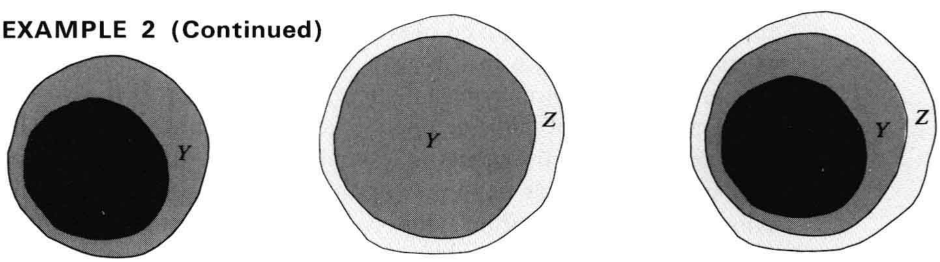
1. All marfets are trilos.
2. No trilos are scarpuls.
3. Some rancels are scarpuls.

a. No rancels are marfets.
b. Some rancels are not marfets.
c. No marfets are scarpuls.

Which, if any, of statements (a), (b), and (c) are inescapable conclusions from statements 1, 2, and 3 taken together as the hypothesis? The student should work this out to his own satisfaction before continuing.

As notation for the study of the logic of sets we denote an arbitrary class or set by a capital letter such as X and any element of the set by the corresponding small letter x. Diagrams representing sets and the relations among them provide an intuitive approach to the rules of reasoning about sets. Represent X by the interior of a closed curve and the elements of X by points within the enclosure. Then relationships among sets involved in statements can be represented by relationships among enclosures. Example 2 then appears as follows.

EXAMPLE 2 (Continued)



All x's are y's   All y's are z's   All x's are z's

Figure 1-1

Since the X enclosure necessarily falls inside the Z enclosure, every point representing an x must be a point representing a z and statement 3 thus becomes an inescapable conclusion (see Figure 1-1).

Such diagrams are called Euler diagrams after the famous Swiss mathematician Leonhard Euler (1707–1783). They are also called Venn diagrams after

John Venn (1834–1923), an English mathematician interested in probability and logic.

At once we ask, "What criterion can be made for the use of Euler diagrams in deductive reasoning?" Intuition about geometric figures leads to some common agreements. First, most of the statements we have made about sets appear in one of the following forms or can be expressed in one of these forms.

A. All x's are y's.
B. No x's are y's.
C. Some x's are y's.
D. Some x's are not y's.

In a given discussion we usually think of each set as a subset of a total set of elements called a *universe* or *population*, denoted by the letter I. In Example 1, rectangles, parallelograms, and polygons may each be thought of as subsets of the set of all plane geometric figures. The elements of a given universe not in a specified set belong to the *complement* of that set. If X denotes a set, then the elements of I not in X are in the complement of X. We denote the complement of X by $\overline{X}$ and an element of $\overline{X}$ by $\bar{x}$. In a diagram the points outside the X enclosure, but inside the universe enclosure, constitute $\overline{X}$, and we call an element $\bar{x}$ in $\overline{X}$ a "not x."

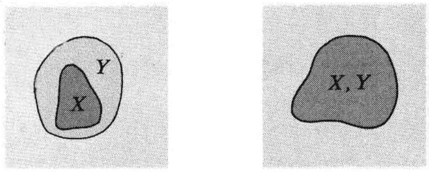
In agreeing on the possible diagrams for statements (A), (B), (C), and (D), we must understand clearly the meanings of the words used in the statements. "All" may be replaced by "every," "each," or "any." "Some" means "at least one and possibly all." This differs from the ordinary use of "some," which does not usually include "possibly all." Statement (B) could be written in the form,

All x's are not y's.

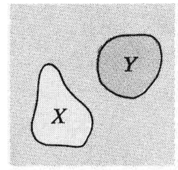
The possible Euler diagrams for statements (A), (B), (C), and (D) appear in Figure 1-2. (See page 6.) In each case the rectangular region represents the universe. In statements (C) and (D) the dot represents the minimal element needed to satisfy the word "some."

Valid reasoning about sets, then, is the process of obtaining those conclusions which are inescapable in the sense that they follow from every possible correct Euler diagram. Such a conclusion is called a *valid conclusion.* Just one correct diagram for which a given conclusion does not hold is enough to show that the reasoning that claims this conclusion to be inescapable is *not valid*; that is, the reasoning is *invalid.*

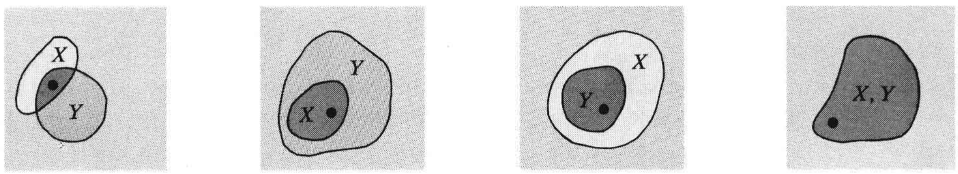
REMARK: The process of drawing Euler diagrams, however, can *never guarantee* that we have found a *valid* conclusion, because we are never sure that every possible diagram has been drawn. The diagrams, however, do aid us in deciding about conclusions that appear to be valid. We can be sure that a conclusion is *not valid* when we have found one diagram that contradicts it.



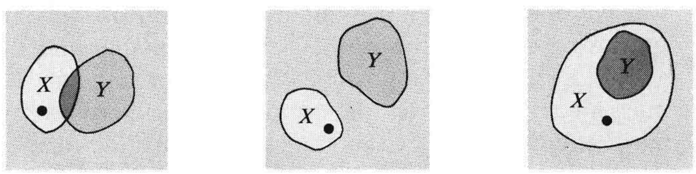
A. All x's are y's



B. No x's are y's



C. Some x's are y's



D. Some x's are not y's

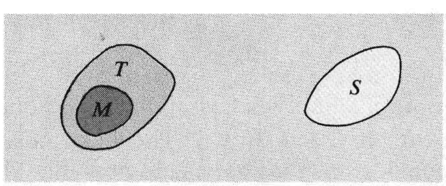
Figure 1-2

EXAMPLE 5

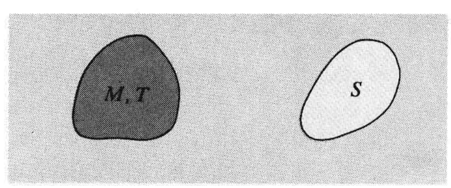
HYPOTHESIS: **1.** All marfets M are trilos T.
2. No trilos are scarpuls S.

CONCLUSION: No marfets are scarpuls.

In both diagrams that may be drawn to fit the hypothesis (Figure 1-3), the M and the S enclosures do not overlap. Hence, the conclusion seems to be *valid.*



Possibility 1



Possibility 2

Figure 1-3