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Preface

Oscillatory and wave phenomena are encountered in almost all branches of physics: mechanics, geophysics, electromagnetism, optics, quantum physics, etc. Some of them were first observed in antiquity, but their scientific study only started in the 17th Century. The phenomena include mechanical vibrations and waves, electromagnetic vibrations and waves, matter waves, etc. Electromagnetic vibrations and waves were discovered in the 19th Century, while matter waves were discovered in the 20th Century. Each branch of physics has its own concepts, and even its own proper mathematical language. Nevertheless, all types of vibrations and waves share several common properties: modes, similar forms of energy, superposition, interference, diffraction, etc.

The purpose of this book is to study oscillatory and wave phenomena at the undergraduate level. It was not conceived with the intended application as a textbook for a specific physics course. Some sections, indicated by an asterisk (*), may prove difficult and may be omitted without loss of continuity.

Chapter 1 introduces the basic concepts and studies some examples of vibrations of mechanical and electromagnetic systems with one or several degrees of freedom. Chapter 2 studies the superposition of vibrations and introduces Fourier analysis. Chapter 3 analyzes forced vibrations and resonances. Chapter 4 introduces the basic notions of waves in infinite media: wave equations and their solutions, energy density and energy transfer, etc. Chapter 5 is devoted to the study of mechanical waves (elastic waves, sound and surface waves). In Chapter 6, we summarize the basic laws of electromagnetism and analyze the electromagnetic waves in insulators, conductors and plasmas. Reflection and refraction are studied in Chapter 7, interference and diffraction are studied in Chapter 8 and finally standing waves and waveguides, in Chapter 9. This book shall not study the emission of waves or optical setups.

The required mathematical techniques are introduced as the need arises. Appendix A aids understanding by summarizing the principal mathematical formulas, integrals and vector analysis. We tried to use clear notations by assigning similar symbols for the various physical quantities: a boldfaced symbol for a vector quantity, an italic symbol for a scalar quantity or a component of a vector quantity, an underlined symbol for a complex quantity, and script symbol for a curve, a surface, a volume and some special quantities. Physical quantities of the same type are referred to by symbols with different indexes: for instance, $f_{(Fr)}, f_{(ez)}, \mathbf{F}_{(E)}$, etc., for the different types of force. The energy is designated by U to avoid confusion with the components of the electric field **E**. The frequency is represented by $\tilde{\nu}$, instead of the usual Greek symbol ν, to avoid its confusion with the velocity v.

A unit vector is often represented by **e**, while the unit vectors of the axes are represented by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_x$. In order to write summations in a condensed form, we sometimes designate the Cartesian coordinates x, y and z by x_1, x_2 and x_3 respectively, and the components of a vector **V** by $V_1 \equiv V_x$, $V_2 \equiv V_y$ and $V_3 \equiv V_z$. The partial derivative of $u(x, y, z, t)$ with respect to time is represented by $\dot{u}$ or $\partial_t u$ and its partial derivatives by $\partial_x u$ for $\partial u/\partial x$, $\partial^2{}_{xt}u$ for $\partial^2 u/\partial x\, \partial t$, etc. We also use the notation $\partial_i u$ for the partial derivatives $\partial u/\partial x_i$ and $\partial_i V_j$ for $\partial V_j/\partial x_i$ (i and $j = 1, 2, 3$).

Each chapter ends with a *Summary* section for the principal results of the chapter, and a section entitled *Problem solving suggestions*, which contains remarks or possible errors to be avoided, approximation methods and further clarifications. For training students, each chapter contains some *examples* that are worked out in detail and two kinds of exercises: *conceptional questions*, a selection of discussion questions designed to develop the understanding of the physical concepts, often without a need for calculations; and *problems*, which are ordered according to the sections of the chapter and arranged in approximate levels of difficulty (an asterisk (*) indicates a problem of some difficulty, two asterisks (**) indicates a problem with some connectional or computational difficulties. The answers to most of the problems are given in a special addendum entitled *Answers to the Problems*, which enables students to check their results.

I hope that this text makes the subject more accessible for students, and that it is utilized as a good teaching aid for professors.

T. BÉCHERRAWY
November 2011

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Chapter 1

Free Oscillations

In this chapter we introduce the basic notions of free oscillations. Starting with a study of the differential equation governing the undamped vibrations, its general solution and its trigonometric, complex and phasor representations, we then progress to the equation of damped oscillations and its solutions. We analyze some simple oscillating systems with one degree of freedom by emphasizing the notion of energy which, in modern physics, is considered to be a more fundamental quantity than forces. We generalize these results to systems undergoing small displacements or variations of the state "back-and-forth" near an equilibrium position. Afterwards, we analyze systems with two or several degrees of freedom.

1.1. Oscillations and waves, period and frequency

Vibrations or *oscillations* are motions or changes in the state of physical systems back-and-forth on both sides of an equilibrium position that are repeated more or less regularly in time. *Waves* are vibrations that propagate from one region to another. We encounter vibratory and wave phenomena in almost all branches of physics: mechanics, geophysics, electromagnetism, optics, quantum physics, etc. We consider in this book two kinds of vibrations: *mechanical vibrations* (of a pendulum, a string, etc.) and *electromagnetic vibrations* (of electric circuits, radio waves, etc.).

Vibrations are *free* if, after an initial excitation, the system oscillates subject to its own internal forces but no-external forces. On the other hand, the vibrations are said to be *forced* if the external force continues to sustain the oscillation of the system. The external force is called the *driving force*.

Vibrations of a system are *periodic* if the system returns exactly to the same state after each time interval T, called the *period* of vibration. Any physical quantity u of the system takes the same value after time intervals T, $2T$, $3T$, etc. Figure 1.1a shows the variation of a periodic function u in time. This periodicity may be expressed mathematically by the relation

$$u(t) = u(t + T) = u(t + 2T) = ... = u(t + nT) \ldots \qquad [1.1]$$

The *frequency* $\tilde{\nu}$ is the number of complete vibrations in unit time, thus

$$\tilde{\nu} = 1/T. \qquad [1.2]$$

In the *International System of Units* (*SI*), the period is expressed in *seconds* (s) and the frequency in s^{-1}, called *hertz* (Hz). For high frequencies, we use *kilohertz* (kHz = 10^3 Hz), *megahertz* (MHz = 10^6 Hz) and *gigahertz* (GHz = 10^9 Hz).

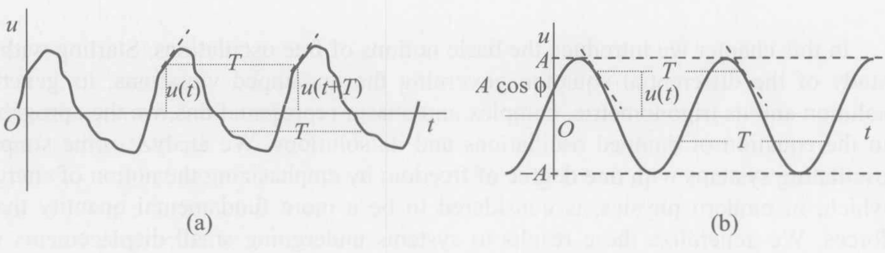

Figure 1.1. *a) Periodic vibration; b) simple harmonic vibration* $u = A\cos(\omega t + \phi)$

1.2. Simple harmonic vibrations: differential equation and linearity

Periodic vibrations are referred to as *harmonic vibrations* by analogy to musical sounds. The simplest periodic vibration is represented by a sine or a cosine function known as a *simple harmonic function*

$$u = A\cos(\omega t + \phi). \qquad [1.3]$$

A is the *amplitude* and ω is the *angular frequency* (often called *frequency* for short). $(\omega t + \phi)$ is the *phase* at time t and ϕ is the *initial phase* (called *phase*, for short). The phase has the dimension of angles, and is therefore expressed in *radians* (rad); while the angular frequency ω is in *radians per second (*rad/s). A and u have the same dimensions, so they are expressed in the same units. The simple harmonic function [1.3] is illustrated in Figure 1.1b.

The values of the sinusoidal function are repeated if the phase varies by 2π (or $2n\pi$ with n as an integer). Thus, u has a period T such that $\omega(t + T) + \phi = \omega t + \phi + 2\pi$; hence, the relationships between ω, T and $\tilde{\nu}$:

$$\omega = 2\pi/T = 2\pi\tilde{\nu} \quad [1.4]$$

By differentiating expression [1.3] twice with respect to time, we obtain:

$$\dot{u} \equiv du/dt = -A\omega \sin(\omega t + \phi) = A\omega \cos(\omega t + \phi + \pi/2) \quad [1.5]$$
$$\ddot{u} \equiv d^2u/dt^2 = -A\omega^2 \cos(\omega t + \phi) = A\omega^2 \cos(\omega t + \phi + \pi). \quad [1.6]$$

Therefore, u is a solution of the differential equation of simple harmonic oscillations

$$\ddot{u} + \omega^2 u = 0. \quad [1.7]$$

This is a second-order homogeneous differential equation. ω is the *natural angular frequency* (also called the *normal angular frequency*). It depends on the physical characteristics of the oscillating system (masses, internal forces, etc.). Expression [1.3] is the *general solution* or the *normal mode* of oscillation. Any oscillation of the free system may be written in this form and any expression of this form is a possible state of oscillation. The *constants of integration* A and ϕ depend on the initial excitation of the system, that is, the *initial conditions* which are the values of u and $\dot{u}$ at a given time t_o. For instance, if the system is set in oscillation at $t_o = 0$, we have the conditions $u(0) = A \cos\phi$ and $\dot{u}(0) = -A\omega \sin\phi$, from which we deduce that

$$A = \sqrt{u(0)^2 + \dot{u}(0)^2/\omega^2}\,, \quad [1.8]$$
$$\cos\phi = u(0)/A, \qquad \sin\phi = -\dot{u}(0)/\omega A\,. \quad [1.9]$$

The relationships in [1.9] determine the phase ϕ. We may also write

$$\tan\phi = -\dot{u}(0)/\omega u(0)\,. \quad [1.10]$$

However, this relationship determines ϕ only up to π. Instead of [1.3], we may use the expression $u = A \sin(\omega t + \phi')$. By adding or subtracting $\pi/2$ or π from the phase, it is possible to write any simple harmonic vibration in the form of [1.3] with a positive amplitude A.

The equation of oscillations [1.7] is *linear* and *homogeneous*. Therefore, it has the important property of satisfying the *superposition principle*: If $u(t)$ is a solution and C is an arbitrary constant, $Cu(t)$ is also a solution. Likewise, if $u_1(t)$ and $u_2(t)$ are two solutions, any linear superposition with arbitrary coefficients C_1 and C_2

$$u(t) = C_1\, u_1(t) + C_2\, u_2(t) \qquad [1.11]$$

is also a solution of the equation. The initial conditions of $u(t)$ are linear combinations of the initial conditions of $u_1(t)$ and $u_2(t)$ with the same coefficients C_1 and C_2. The superposition principle plays a crucial role in many branches of physics.

Equation [1.7] may be written in the form $D.u = 0$, where $D = d^2/dt^2 + \omega^2$ or a more complicated form. D is called an *operator*. It transforms a function u to another function $D.u$. If D verifies the condition $D.(C_1u_1 + C_2u_2) = C_1\, D.u_1 + C_2\, D.u_2$, it is said to be *linear*. If u_1 and u_2 are two solutions of the linear and homogeneous differential equation, $D.u = 0$, any linear superposition $u = C_1u_1 + C_2u_2$ is also a solution, for any values of the constants C_1 and C_2. If u_1 and u_2 are two independent solutions of the second-order differential equation $Du = 0$, u is the general solution of this equation, because it depends on two arbitrary and independent constants C_1 and C_2. A differential equation $D.u = f$ is not homogeneous because it contains a term f that is independent of u.

The general solution of differential equation [1.7] may be written in one of the following equivalent forms

$$u(t) = A\cos(\omega t + \phi) \qquad [1.12]$$
$$= A'\sin(\omega t + \phi') \qquad [1.13]$$
$$= A_1\cos(\omega t) + A_2\sin(\omega t). \qquad [1.14]$$

Each of these expressions depends on two independent parameters that are determined by the initial conditions. The relationships between these parameters are:

$$A' = A \text{ and } \phi' = \phi + \pi/2 \qquad [1.15]$$
$$A_1 = A\cos\phi = A\sin\phi', \qquad A_2 = -A\sin\phi = A\cos\phi'. \qquad [1.16]$$

Note that, in form [1.14], A_1 and A_2 may be positive, negative or zero. If the amplitudes A and A' are chosen to be positive, we have the following relationships:

$$A = \sqrt{A_1^{\,2} + A_2^{\,2}}, \qquad \tan\phi = -A_2/A_1, \qquad \tan\phi' = A_1/A_2. \qquad [1.17]$$

EXAMPLE 1.1. Write the expression $u = 3\cos(\omega t) + 2\cos(\omega t - \pi/3)$ in the form $u = A\cos(\omega t + \phi)$.

SOLUTION – Using the addition formula for $\cos(\omega t - \pi/3)$, we find

$$u = 3\cos(\omega t) + 2\cos\frac{\pi}{3}\cos(\omega t) + 2\sin\frac{\pi}{3}\sin(\omega t) = 4\cos(\omega t) + 1.732\sin(\omega t).$$

We may write u in the form $A\cos(\omega t + \phi) = A\cos\phi\cos(\omega t) - A\sin\phi\sin(\omega t)$ if $A\cos\phi = 4$ and $A\sin\phi = -1.732$. By squaring both sides of these equations and adding them, we find $A^2 = 4^2 + 1.732^2$. Hence $A = 4.359$, $\cos\phi = 0.9177$ and $\sin\phi = -0.3974$. Thus we deduce that $\phi = -0.4086$ rad and $x = 4.359\cos(\omega t - 0.4086)$.

1.3. Complex representation and phasor representation

A *complex variable* (designated by an underlined symbol) may be written in the algebraic form

$$\underline{z} = x + \mathrm{i}y, \qquad \text{with} \qquad x = \mathcal{R}e\,\underline{z} \qquad \text{and} \qquad y = \mathcal{I}m\,\underline{z}, \qquad [1.18]$$

where $\mathrm{i}^2 \equiv -1$. The *real part* of $\underline{z}$ is x and its *imaginary part* is y. A complex number $\underline{z}$ is usually represented by a point of coordinates x and y in the *Oxy* plane called an *Argand diagram* (Figure 1.2a). We may also use the exponential form in terms of the polar coordinates

$$\underline{z} = \rho\cos\phi + \mathrm{i}\,\rho\sin\phi = \rho\, e^{\mathrm{i}\phi}, \qquad \text{where } \rho \equiv |\underline{z}| = \textit{modulus}\ \underline{z} \qquad \text{and } \phi = \textit{phase}\ \underline{z}. \quad [1.19]$$

$\rho \equiv |\underline{z}|$ is the *modulus* of $\underline{z}$ and ϕ is the *phase* of $\underline{z}$, where we have used Euler equation (see section A.5 of Appendix A)

$$e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi. \qquad [1.20]$$

The two representation are related by the equations

$$x = \rho\cos\phi, \quad y = \rho\sin\phi; \quad \rho = \sqrt{x^2 + y^2}, \quad \cos\phi = x/\rho, \quad \sin\phi = y/\rho. \qquad [1.21]$$

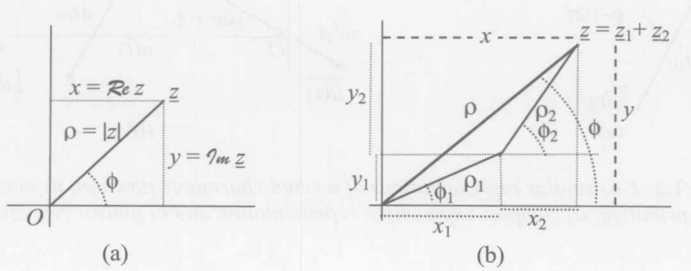

Figure 1.2. *a) Argand diagram; b) sum of two complex numbers* $\underline{z}_1$ *and* $\underline{z}_2$

The sum of two complex numbers is easily evaluated using the algebraic form (Figure 1.2b)

$$\underline{z}_1 + \underline{z}_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \qquad [1.22]$$

while their product and ratio are easily evaluated using the exponential form:

$$\underline{z}_1 \underline{z}_2 = (\rho_1 e^{i\phi_1})(\rho_2 e^{i\phi_2}) = \rho_1 \rho_1 e^{i(\phi_1+\phi_2)} \qquad [1.23]$$

$$\underline{z}_1/\underline{z}_2 = (\rho_1 e^{i\phi_1})/(\rho_2 e^{i\phi_2}) = (\rho_1/\rho_1)\ e^{i(\phi_1-\phi_2)}. \qquad [1.24]$$

It is easy to verify that differential equation [1.7] has the *general complex solution*

$$\underline{u}(t) = \underline{C}\, e^{i\omega t} \qquad \text{with} \qquad \underline{C} \equiv C e^{i\alpha}, \qquad [1.25]$$

where $\underline{C}$ is the *complex amplitude* and $C = |\underline{C}|$ is its modulus. This expression depends on two real parameters C and α, as it should for any general solution of a second order differential equation. By taking the real part, we find

$$u(t) = \mathcal{Re}\, \underline{u}(t) = \mathcal{Re}\, [C e^{i(\omega t + \alpha)}] = C \cos(\omega t + \alpha). \qquad [1.26]$$

Comparing this with the expression $u = A \cos(\omega t + \phi)$, we deduce that

$$A = |\underline{C}| = \textit{modulus}\ \underline{C}, \qquad \phi = \alpha = \textit{phase}\ \underline{C}. \qquad [1.27]$$

Thus, the amplitude and the phase of the real solution u are respectively the modulus and the phase of the complex amplitude $\underline{C}$ of the complex solution.

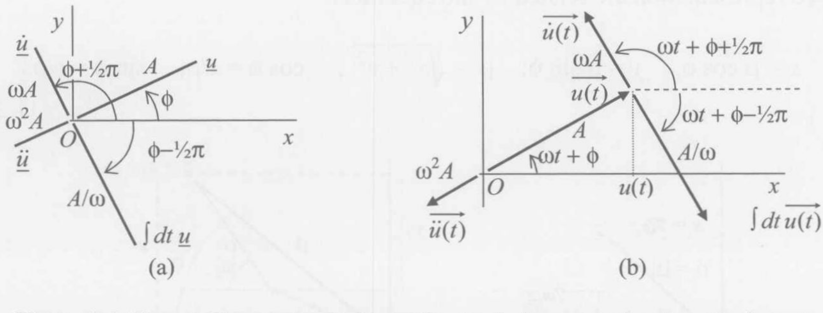

Figure 1.3. *Two similar representations of a simple harmonic function, its derivatives and its primitive: a) complex exponential representation and b) phasor representation*