Calculus

With Analytic Geometry

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To my wife Shirley, and our children,
Mary, Mark, John, Steven, Paul, Thomas, Robert.

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Table of Integrals

Basic Forms

$$\int u\,dv = uv - \int v\,du$$

$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{1}{\ln a}a^u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int \tan u\,du = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

Forms Involving $\sqrt{a^2 + u^2}$

$$\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln|u + \sqrt{a^2 + u^2}| + C$$

$$\int u^2\sqrt{a^2 + u^2}\,du = \frac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln|u + \sqrt{a^2 + u^2}| + C$$

$$\int \frac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$$

$$\int \frac{\sqrt{a^2 + u^2}}{u^2}\,du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln|u + \sqrt{a^2 + u^2}| + C$$

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \ln|u + \sqrt{a^2 + u^2}| + C$$

$$\int \frac{u^2\,du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\sqrt{a^2 + u^2} - \frac{a^2}{2}\ln|u + \sqrt{a^2 + u^2}| + C$$

$$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$$

$$\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2u} + C$$

$$\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$$

Forms Involving $\sqrt{a^2 - u^2}$

$$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

$$\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$$

$$\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$$

$$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2u}\sqrt{a^2 - u^2} + C$$

$$\int (a^2 - u^2)^{3/2}\,du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$$

Forms Involving $\sqrt{u^2 - a^2}$

$$\int \sqrt{u^2 - a^2}\,du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int u^2\sqrt{u^2 - a^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{u} + C$$

$$\int \frac{\sqrt{u^2 - a^2}}{u^2}\,du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$$

$$\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$$

Forms Involving $a + bu$

$$\int \frac{u\,du}{a + bu} = \frac{1}{b^2}(a + bu - a\ln|a + bu|) + C$$

$$\int \frac{u^2\,du}{a + bu} = \frac{1}{2b^3}[(a + bu)^2 - 4a(a + bu) + 2a^2\ln|a + bu|] + C$$

$$\int \frac{du}{u(a + bu)} = \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C$$

$$\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$$

$$\int \frac{u\,du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2}\ln|a + bu| + C$$

$$\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$$

$$\int \frac{u^2\,du}{(a + bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a\ln|a + bu|\right) + C$$

$$\int u\sqrt{a + bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a + bu)^{3/2} + C$$

$$\int \frac{u\,du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu}$$

$$\int \frac{u^2\,du}{\sqrt{a + bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a + bu}$$

$$\int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C, \quad \text{if } a > 0$$

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a + bu}{-a}} + C, \quad \text{if } a < 0$$

$$\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}}$$

$$\int \frac{\sqrt{a + bu}}{u^2}\,du = -\frac{\sqrt{a + bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a + bu}}$$

$$\int u^n\sqrt{a + bu}\,du = \frac{2u^n(a + bu)^{3/2}}{b(2n + 3)} - \frac{2na}{b(2n + 3)}\int \frac{u^{n-1}}{\sqrt{a + bu}}\,du$$

$$\int \frac{u^n\,du}{\sqrt{a + bu}} = \frac{2u^n\sqrt{a + bu}}{b(2n + 1)} - \frac{2na}{b(2n + 1)}\int \frac{u^{n-1}\,du}{\sqrt{a + bu}}$$

$$\int \frac{du}{u^n\sqrt{a + bu}} = -\frac{\sqrt{a + bu}}{a(n - 1)u^{n-1}} - \frac{b(2n - 3)}{2a(n - 1)}\int \frac{du}{u^{n-1}\sqrt{a + bu}}$$

Trigonometric Forms

$$\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$$

$$\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$$

$$\int \tan^2 u\,du = \tan u - u + C$$

$$\int \cot^2 u\,du = -\cot u - u + C$$

$$\int \sin^3 u\,du = -\tfrac{1}{3}(2 + \sin^2 u)\cos u + C$$

$$\int \cos^3 u\,du = \tfrac{1}{3}(2 + \cos^2 u)\sin u + C$$

$$\int \tan^3 u\,du = \tfrac{1}{2}\tan^2 u + \ln|\cos u| + C$$

$$\int \cot^3 u\,du = -\tfrac{1}{2}\cot^2 u - \ln|\sin u| + C$$

$$\int \sec^3 u\,du = \tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u| + C$$

$$\int \csc^3 u\,du = -\tfrac{1}{2}\csc u\cot u + \tfrac{1}{2}\ln|\csc u - \cot u| + C$$

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n - 1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n - 1}{n}\int \cos^{n-2} u\,du$$

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$$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$$

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$$

$$\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$

$$\int u \cos u\,du = \cos u + u \sin u + C$$

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$$

$$\int \sin^n u \cos^m u\,du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du$$
$$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du$$

$$\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$$

$$\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$$

$$\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int u \sin u\,du = \sin u - u\cos u + C$$

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$$

Inverse Trigonometric Forms

$$\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$$

$$\int \tan^{-1} u\,du = u\tan^{-1} u - \tfrac{1}{2}\ln(1+u^2) + C$$

$$\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u^n \sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$$

$$\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$$

$$\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$$

Exponential and Logarithmic Forms

$$\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$$

$$\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$$

$$\int \ln u\,du = u\ln u - u + C$$

$$\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$$

$$\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

$$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$$

$$\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$$

Hyperbolic Forms

$$\int \sinh u\,du = \cosh u + C$$

$$\int \tanh u\,du = \ln\cosh u + C$$

$$\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$$

$$\int \operatorname{sech}^2 u\,du = \tanh u + C$$

$$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$$

$$\int \cosh u\,du = \sinh u + C$$

$$\int \coth u\,du = \ln|\sinh u| + C$$

$$\int \operatorname{csch} u\,du = \ln\left|\tanh \tfrac{1}{2}u\right| + C$$

$$\int \operatorname{csch}^2 u\,du = -\coth u + C$$

$$\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$$

Forms Involving $\sqrt{2au - u^2}$

$$\int \sqrt{2au - u^2}\, du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int u\sqrt{2au - u^2}\, du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{\sqrt{2au - u^2}}{u}\, du = \sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{\sqrt{2au - u^2}}{u^2}\, du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{u\, du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{u^2\, du}{\sqrt{2au - u^2}} = -\frac{(u + 3a)}{2}\sqrt{2au - u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{2ua - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$$

Preface

Most students study calculus in order to use it as a tool in areas other than mathematics. They desire information about *why* calculus is important, and *where* and *how* it can be applied. As I wrote this text, I tried to keep these facts in mind. In particular, before an important concept is defined, problems which require the concept are presented. After sufficient theory has been developed, there are many interesting physical and mathematical examples to draw upon. However, the difficulty is to arouse student interest at the *beginning* of a new subject.

To illustrate my approach to calculus, in this text the limit concept is motivated by referring to three practical problems, one from physics, another from chemistry, and the third from mathematics. The notion of limit is then discussed in an intuitive manner, using numerical examples. A precise definition is introduced a section later, but only after references are made to previous examples. The definition is then reinforced through the use of two different graphical techniques. I believe that students should not spend an entire semester or more repeating the words "closer and closer," nor should they be literally buried under epsilons and deltas! Limit theorems are stated and used in examples, but difficult proofs are placed in an appendix, where they may be left as reading assignments, discussed immediately, or postponed until a later time. A similar philosophy is followed when the derivative, the definite integral, and other important concepts are introduced.

In addition to achieving a good balance between rigor and intuition, my primary objective was to write a book which could be read and understood by college freshmen. Throughout each section I have striven for accuracy and clarity of exposition, together with a presentation which makes the transition from precalculus mathematics to calculus as smooth as possible.

This text contains sufficient material for any of the standard calculus sequences. The Table of Contents shows the order in which the material is presented. In general, Chapters 1 through 6 could constitute the equivalent of a one-semester course for students who only need a basic background consisting of limits, derivatives, and definite integrals of algebraic functions. Chapters 7 through 12 would ordinarily make up the second semester of work; however, Chapter 12 on infinite series could be postponed until the third semester. In this event, Chapter 13 on curves and polar coordinates, or parts of Chapter 14 on vectors could be substituted. The remainder of the text is intended for what is usually referred to as the third semester. Chapter 18 on vector calculus is somewhat unusual for a first course. Some instructors may wish to include this material and others not. For this reason it is placed near the end of the book, where portions may be omitted without interrupting the continuity of the text. The same is true for Chapter 19 on differential equations.

A great deal of thought was given to the construction of exercise sets. There are over 4,000 exercises, enough to keep even the most industrious student busy! Many are of the drill variety and should be attempted by everyone. Others are challenging and are intended for more highly motivated students. There is a review section at the end of each chapter consisting of a list of important topics together with pertinent exercises. The review exercises are similar in scope to those which appear throughout the chapter and may be used by students to prepare for examinations.

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Earl W. Swokowski

Other books by Earl W. Swokowski:

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Introduction

What is Calculus?

Calculus was invented in the seventeenth century to provide a tool for solving problems involving motion. The subject matter of geometry, algebra, and trigonometry is applicable to objects which move at constant speeds; however, methods introduced in calculus are required to study the orbits of planets, to calculate the flight of a rocket, to predict the path of a charged particle through an electromagnetic field and, for that matter, to deal with all aspects of motion.

In order to discuss objects in motion it is essential first to define what is meant by *velocity* and *acceleration.* Roughly speaking, the velocity of an object is a measure of the rate at which the distance traveled changes with respect to time. Acceleration is a measure of the rate at which velocity changes. Velocity may vary considerably, as is evident from the motion of a drag-strip racer or the descent of a space capsule as it reenters the Earth's atmosphere. In order to give precise meanings to the notions of velocity and acceleration it is necessary to use one of the fundamental concepts of calculus, the *derivative.*

Although calculus was introduced to help solve problems in physics, it has been applied to many different fields. One of the reasons for its versatility is the fact that the derivative is useful in the study of rates of change of many entities other than objects in motion. For example, a chemist may use derivatives to forecast the outcome of various chemical reactions. A biologist may employ it in the investigation of the rate of growth of bacteria in a culture. An electrical engineer uses the derivative to describe the change in current in an electric circuit. Economists have applied it to problems involving corporate profits and losses.

The derivative is also used to find tangent lines to curves. Although this has some independent geometric interest, the significance of tangent lines is of major importance in physical problems. For example, if a particle moves along a curve, then the tangent line indicates the direction of motion. If we restrict our attention to a sufficiently small portion of the curve, then in a certain sense the tangent line may be used to approximate the position of the particle.

Many problems involving maximum and minimum values may be attacked with the aid of the derivative. Some typical questions that can be answered are: At what angle of elevation should a projectile be fired in order to achieve its maximum range? If a tin can is to hold one gallon of a liquid, what dimensions require the least amount of tin? At what point between two light sources will the illumination be greatest? How can certain corporations maximize their revenue? How can a manufacturer minimize the cost of producing a given article?

Another fundamental concept of calculus is known as the *definite integral.* It, too, has many applications in the sciences. A physicist uses it to find the work required to stretch or compress a spring. An engineer may use it to find the center of mass or moment of inertia of a solid. The definite integral can be used by a biologist to calculate the flow of blood through an arteriole. An economist may employ it to estimate depreciation of equipment in a manufacturing plant. Mathematicians use definite integrals to investigate such concepts as areas of surfaces, volumes of geometric solids, and lengths of curves.

All the examples we have listed, and many more, will be discussed in detail as we progress through this book. There is literally no end to the applications of calculus. Indeed, in the future perhaps *you*, the reader, will discover new uses for this important branch of mathematics.

The derivative and the definite integral are defined in terms of certain limiting processes. The notion of limit is the initial idea which separates calculus from the more elementary branches of mathematics. Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) discovered the connection between derivatives and integrals. Because of this, and their other contributions to the subject, they are credited with the invention of calculus. Many other mathematicians have added a great deal to its development.

The preceding discussion has not answered the question "What is calculus?" Actually, there is no simple answer. Calculus could be called the study of limits, derivatives, and integrals; however, this statement is meaningless if definitions of the terms are unknown. Although we have given a few examples to illustrate what can be accomplished with derivatives and integrals, neither of these concepts has been given any meaning. Defining them will be one of the principal objectives of our early work in this text.