TRIGONOMETRY

A MODERN APPROACH

TRIGONOMETRY

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OMETRY

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Rules of Exponents

$$a^{x+y} = a^x a^y$$

$$(ab)^x = a^x b^x$$

$$(a^x)^y = a^{xy}$$

$$a^{-x} = \frac{1}{a^x}$$

Rules of Logarithms

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^r = r \log_a x$$

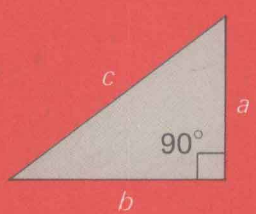
$$a^{\log_a x} = x, \quad \log_a a^x = x$$

Quadratic Formula

$$ax^2 + bx + c = 0 \text{ has roots } r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \; a \neq 0$$

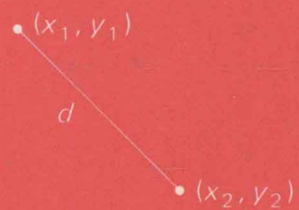
$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

Pythagorean Formula



$$a^2 + b^2 = c^2$$

Distance Formula



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Constants

$\pi \approx 3.14159$

1 radian $= \dfrac{180}{\pi}$ degrees $\approx 57.29578^\circ$

1 degree $= \dfrac{\pi}{180}$ radians $\approx .017453$ radians

$\dfrac{\pi}{6}$ radians $= 30^\circ$

$\dfrac{\pi}{4}$ radians $= 45^\circ$

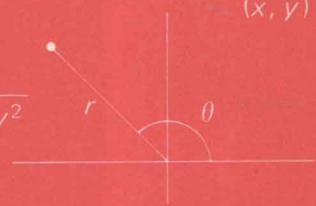
$\dfrac{\pi}{3}$ radians $= 60^\circ$

$\dfrac{\pi}{2}$ radians $= 90^\circ$

Complex Numbers

$$(x, y) = (r\cos\theta, r\sin\theta)$$
$$= r\cos\theta + ir\sin\theta$$

$$r^2 = \sqrt{x^2 + y^2}$$



$$(a + bi) + (c + di) = (a + d) + i(b + d)$$

$$(a + bi)(c + di) = (ac - bd) + i(ad + bc)$$

$$\overline{a + bi} = a - bi$$

$$|a + bi| = \sqrt{a^2 + b^2}$$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta$ De Moivre's Theorem

The nth roots of $r(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad k = 0, 1, 2, \ldots, n - 1$$

TRIGONOMETRY

PREFACE

A premise underlying this book is that pocket calculators have rendered the traditional logarithmic calculations found in trigonometric texts obsolete. Therefore the calculational methods used in this book assume that all students have access to a calculator that will do either trig functions or basic arithmetic computations.

There are sections included on using tables and doing linear interpolation. Although some of this material should be taught to students having calculators that handle the trig functions, the instructor has the option of determining how much of this material should be covered.

For those courses in which the students have calculators doing only the basic arithmetic computations, the table of functional values at the back of the book will suffice. There are exercises scattered throughout the text which necessitate a high-powered calculator, but these are identified, and they can be omitted with impunity.

It is worth mentioning that the book can also be used by students without calculators of any kind if the students do multiplication and division by hand. The section on logarithmic computation (§27) can be exploited to do such computations with the aid of tables.

I believe that the sequence of topics I have chosen is pedagogically superior to the one usually found in trigonometry texts. The sequence of topics reflects the historical development of trigonometry. It proceeds from the specific (that is, triangles) to the abstract in step-by-step stages. Each step is illustrated and motivated by concrete, realistic applications and problems whenever feasible.

The basic core of the text is found in Chapters 2 and 3. The material in §§1, 2 and 3 of Chapter 1 is prerequisite to this discussion. Chapter 4, which is essentially independent of Chapters 2 and 3, and Chapter 5, which is dependent on Chapter 3 but not on 4, are supplementary to the core chapters.

Most sections can be covered in one class session. However, there are some sections which require more time to cover in detail. These latter sections form a logical unit which is difficult to subdivide.

I have tried to make the book flexible, not only in respect to the accessibility of calculators but also in regard to the depth the instructor wishes to teach. Proofs of theorems and formulas, as well as the discussion of the physics of waves, are isolated and can be covered quickly or even omitted. The examples and exercises have been ordered linearly from simple to difficult.

I would like to thank the following individuals who made valuable suggestions for the improvement of this text: David Belanger, University of South Alabama; James Brady, Santa Barbara City College; Rodney Chase, Oakland Community College; Joseph Dorsett, St. Petersburg Junior College; Samuel Gale, Kingsborough City College; Stuart Goldenberg, California Polytechnic State University, San Luis Obispo; Richard Kennedy, Central Connecticut State College; Victor Klee, University of Washington; Gary Lippmann, California State University, Hayward; R.M. Mallen, Santa Barbara City College; Wayne McDaniel, University of Missouri, St. Louis; Louis Perone, SUNY, Farmingdale; Robert Stephens, Moorpark College; Robert Troyer, Lake Forest College; William Vick; Broome Community College; Abraham Weinstein, Nassau Community College.

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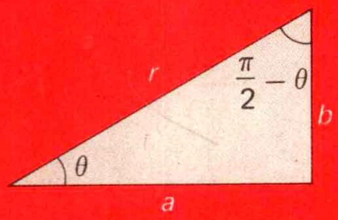
JACK CEDER

Right Triangle Trigonometry



$$\sin\theta = \frac{b}{r} = \cos\left(\frac{\pi}{2} - \theta\right)$$

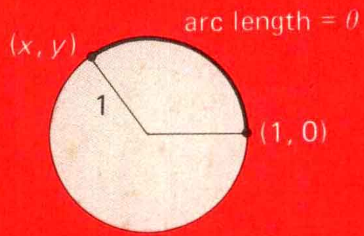
$$\cos\theta = \frac{a}{r} = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \frac{b}{a} = \cot\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \frac{r}{a} = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\theta = \frac{r}{b} = \sec\left(\frac{\pi}{2} - \theta\right)$$

Circular Functions

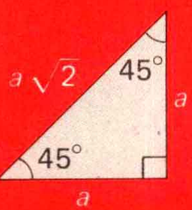


$$\cos\theta = x \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\sin\theta = y \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta}$$

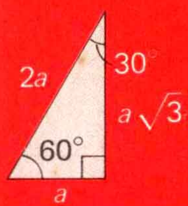
Special Trangles



$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = 1$$



$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\tan 60^\circ = \sqrt{3}$$

Inverse Trigonometric Functions

$$\text{If } \left\{\begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant x \leqslant 1 \\ x \text{ is real} \end{matrix}\right\}, \text{ then } \left\{\begin{matrix} \arccos x \\ \arcsin x \\ \arctan x \end{matrix}\right\} \text{ is that unique number } \theta$$

$$\text{satisfying } \left\{\begin{matrix} 0 \leqslant \theta \leqslant \pi \\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{matrix}\right\} \text{ for which } \left\{\begin{matrix} \cos\theta = x \\ \sin\theta = x \\ \tan\theta = x \end{matrix}\right\}.$$

Common Formulas

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin^2\frac{\theta}{2} = \frac{1 - \cos\alpha}{2}$$

$$\sec^2\theta = 1 + \tan^2\theta$$

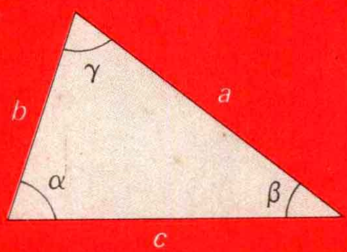
$$\sin(-\theta) = -\sin\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\cos^2\frac{\theta}{2} = \frac{1 + \cos\alpha}{2}$$

$$\csc^2\theta = 1 + \cot^2\theta$$

$$\cos(-\theta) = \cos\theta$$



$$\alpha + \beta + \gamma = \pi$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

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CHAPTER 1

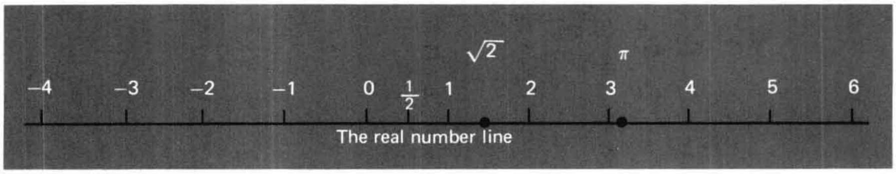
preliminaries

§ 1 numbers, distances, and graphs

This chapter covers some of the more important mathematical preliminaries for the study of trigonometry. Most of this material should be review for those of you who have already studied some algebra. Therefore our analysis is brief, incorporating only notations and terminology necessary for the discussion in this chapter.

the real line • We assume that you are familiar with real numbers, together with their arithmetic properties, so we need only mention some important points here. First of all, by the *real number line* we understand any line coordinatized so that each point in the line corresponds to exactly one real number, and conversely. See Figure 1-1.

Figure 1-1



As real numbers we understand the set of all infinite decimals, both positive and negative. Each decimal is unique except that one ending in a tail of 9's also can be represented as a decimal with a tail of 0's. For example, $4.000\ldots = 3.999\ldots$.

There are several types of real numbers that are important enough to be distinguished. One type consists of those numbers which have a repeating block of digits in their decimals such as $.419419419\ldots$ or $.431292929\ldots$. Such a real number is called a *rational number,* and it can also be represented as a quotient of integers. For example,

$$4.333\ldots = \frac{13}{3},$$

$$.419419\ldots = \frac{419}{999}.$$

An *integer* is a rational number of the form 0, ± 1, ± 2, ± 3, and so forth. The decimal expansion of an integer has nothing but 0's after the decimal point. For example, $3 = 3.000\ldots$.

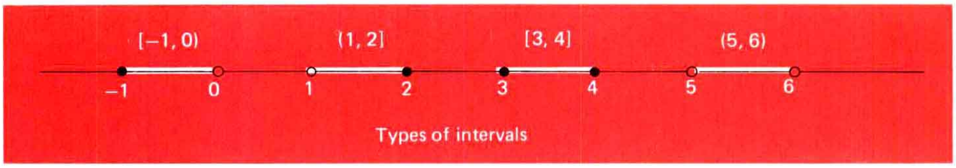
intervals • A subset of real numbers having the property that all numbers between any two members of the set also belong to the set is called an *interval.* There are several types of intervals. If a and b are real numbers with $a \leqslant b$, we define*

*The notation $\{x:\Phi\}$ where Φ is a statement involving x means "the set of all x for which Φ is true."

$$
\begin{aligned}
(a,b) &= \{x : a < x < b\} && \text{an open interval,}\\
[a,b] &= \{x : a \leqslant x \leqslant b\} && \text{a closed interval,}\\
[a,b) &= \{x : a \leqslant x < b\} && \left\{\begin{array}{l}\text{a half-open or}\\ \text{half-closed interval.}\end{array}\right.\\
(a,b] &= \{x : a < x \leqslant b\} &&
\end{aligned}
$$

The parenthesis in the interval notation indicates the exclusion of the point whereas the square bracket indicates inclusion. See Figure 1-2.

Figure 1-2



planar coordinates • If we take two mutually perpendicular real number lines, as illustrated in Figure 1-3, we obtain the *Cartesian coordinate system.* The horizontal and vertical lines are called the x-axis and y-axis, respectively.

This coordinate system allows us to establish a natural one-to-one correspondence between points in the plane and pairs of real numbers as follows. Consider a point P in the plane. Through it draw lines parallel to the two coordinate axes. The number a on the x-axis at which the vertical line intersects the x-axis is called the *x-coordinate* of P. The number b on the y-axis at which the horizontal line intersects the y-axis is called the *y-coordinate* of P. We associate the pair of numbers (a,b) with the point P.* If we reverse this process, we can associate a point P in the plane with each pair of numbers (a,b). See Figure 1-3.

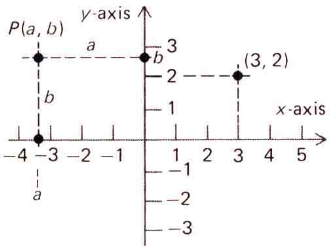
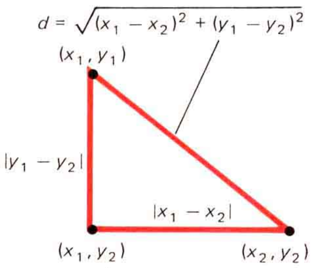


Figure 1-3

distance • The distance between two points on the real number line or, equivalently, two real numbers is the absolute value of their difference. For example, the distance between 4 and -3 is $|+4-(-3)| = |7| = 7$. See Figure 1-4. In general, the distance between real numbers x_1 and x_2 is $|x_1 - x_2|$.



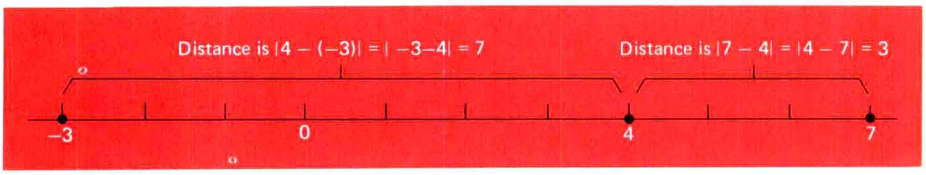


Figure 1-4

Let us now find the distance between two points of the plane. Assuming that the two points (x_1,y_1) and (x_2,y_2) are not on the same vertical line, we may form a right triangle as depicted in Figure 1-5.

The side lengths of this triangle are $|x_1 - x_2|$ and $|y_1 - y_2|$. Therefore the distance d between the points is given by the Pythagorean Theorem (that is, the sum of the squares of the side lengths equals the square of the hypotenuse length):

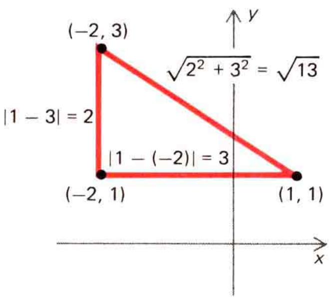


Figure 1-5

*We have used (a,b) in two senses: a point in the plane and an open interval. This ambiguous notation is unfortunately widespread. However, no confusion is likely because the context will tell you which of the two meanings is meant.

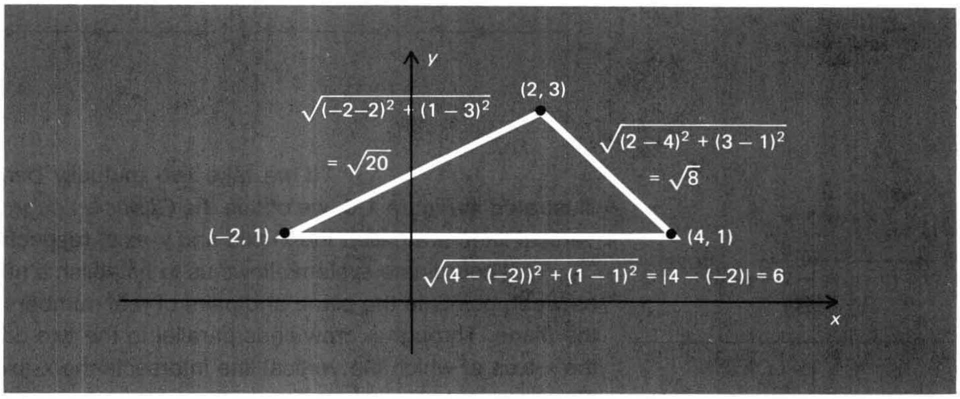
$$d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

In other words,

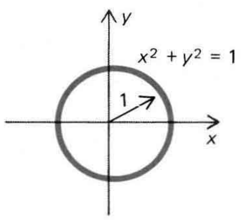
$$\boxed{\text{the distance between } (x_1,y_1) \text{ and } (x_2,y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.}$$

Figure 1-6 illustrates this distance formula applied to a specific triangle.

Figure 1-6



Note that this formula also holds when the two points are on the same vertical line, that is, $x_1 = x_2$. In this case the distance is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(y_1 - y_2)^2} = |y_1 - y_2|$. See Figure 1-5.



The unit circle

example 1-1 Let us find the equation of the circle of radius 1 centered at (0,0).

solution: If (x,y) is any point on the circle, its distance from (0,0) is

$$1 = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

by applying the distance formula. Squaring both sides, we obtain

$$1 = x^2 + y^2.$$

Conversely, if (x,y) satisfies the equation $x^2 + y^2 = 1$, it also satisfies $\sqrt{x^2 + y^2} = 1$ and $\sqrt{(x - 0)^2 + (y - 0)^2} = 1$ so that its distance from (0,0) is 1; that is, it lies on the circle.

Therefore

$$1 = x^2 + y^2$$

is the equation of the circle of radius 1 centered at (0,0). This circle is called the *unit circle.* See Figure 1-7.

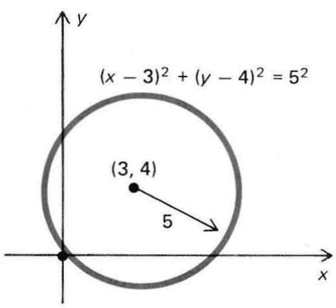


Figure 1-7

example 1-2 Let us find the equation of the circle of radius 5 centered at (3,4).

solution: The condition that a point (x,y) be of distance 5 from (3,4) is

$$5 = \sqrt{(x-3)^2 + (y-4)^2}$$

or

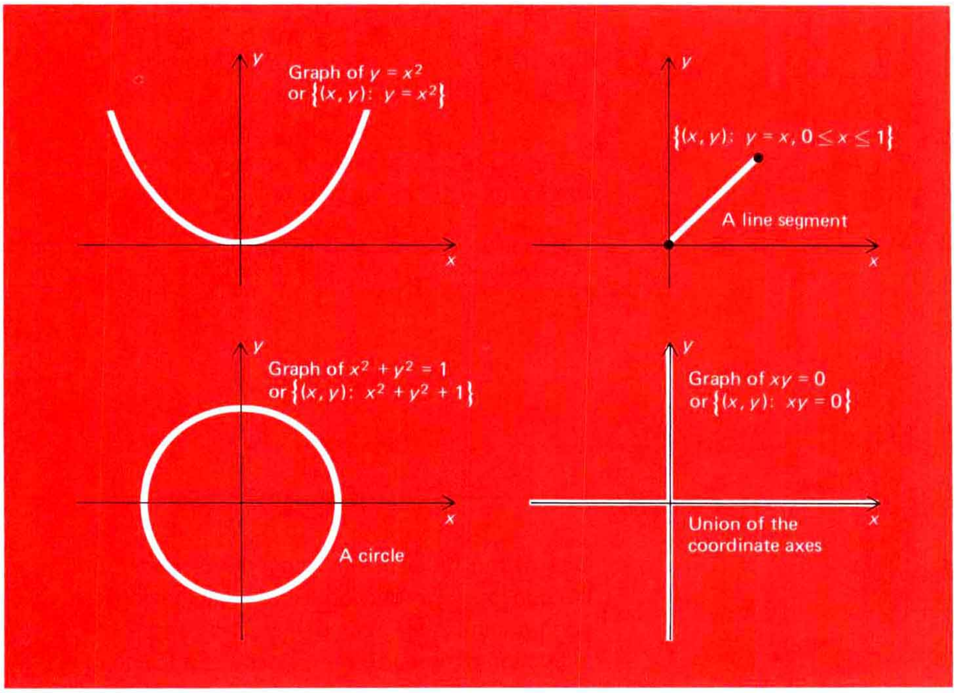
$$(x-3)^2 + (y-4)^2 = 5^2.$$

Conversely, as argued in the solution of Example 1-1, any point (x,y) satisfying this equation must lie on the desired circle. See Figure 1-7.

---

graphs of equations • We can use the planar coordinate system to pictorially illustrate the relationships of the values x and y in an equation involving x and y. Specifically, the *graph* of an equation in x and y consists of all points (x,y) satisfying the equation. Figure 1-8 gives examples of typical graphs.

Figure 1-8



EXERCISES

In Exercises 1-1 through 1-6 write the given set in interval notation.

1-1. the intersection of $[1,5)$ and $[2,6]$, that is, the set of numbers common to both sets
1-2. the intersection of $[-2,3)$ and $(0,3]$
1-3. the union of $[1,5)$ and $[2,6]$, that is, the set of numbers in one or both of the sets
1-4. the union of $(3,6]$ and $(4,5]$
1-5. $\{x : x^2 < 2\}$

1-6. $\{x : x^2 \leq 3\}$
1-7. Plot the following points in the same coordinate system: (1,0), (2,5) (−1,3), (4,2), (−3,−3), and (−1,5).

In Exercises 1-8 through 1-26 sketch the set of points (x,y) in the plane satisfying the given conditions or equations.

1-8. $x = 2$
1-9. $y = -1$
1-10. $xy = 0$ and $x \geq 0$
1-11. $xy = 1$
1-12. $-2 \leq x \leq 2$
1-13. $0 \leq y \leq 1$
1-14. $0 < x \leq 3$
1-15. $0 < x$ and $y \leq 3$
1-16. $x^2 > 3$
1-17. $xy > 0$
1-18. x is an integer
1-19. $x^2 + y^2 \leq 0$
1-20. $x = x^3$
1-21. $y = 2x, \quad -1 \leq x \leq 1$
1-22. $y = \frac{x}{2}, \quad 0 < x < 2$
1-23. $y = x^2, \quad 0 \leq x \leq 2$
1-24. $y^2 = x, \quad x \leq 4$
1-25. $(x-1)^2 + y^2 = 4$
1-26. $(x-1)^2 + (y-2)^2 = 9$

In Exercises 1-27 through 1-30 find the lengths of the sides of the triangles determined by the three given points. Also, draw the triangle.

1-27. (−1,2) and (5,3) and (0,0)
1-28. (1,3) and (2,7) and (1,1)
1-29. (−1,−2) and (−1,3) and (2,−2)
1-30. (0,5) and (2,4) and (3,4)

In Exercises 1-31 through 1-34 find the equation of the circle with the given radius and center. Use Example 1-2 as a model, and draw the graph.

1-31. radius = 2, center at (0,0)
1-32. radius = 3, center at (0,1)
1-33. radius = 4, center at (−2,3)
1-34. radius = r, center at (a,b)

§2 *functions*

One of the most useful concepts of mathematical analysis, including trigonometry, is the concept of a function. Roughly speaking, a function is a dependence of one quantity upon another. For example, we may speak of (1) the population of the United States as a function of time, (2) the velocity of a falling object as a function of time, (3) the price of a commodity as a function of supply and demand.

For the purpose of this book it will suffice* to define a *function* as

> *a rule which associates to each number in a certain set (called the domain) exactly one number.*

*In more abstract mathematics we need to be more precise and logical in the definition of a function.