COMPLEX VARIABLES AND THE LAPLACE TRANSFORM FOR ENGINEERS

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Department of Electrical and Computer Engineering
Syracuse University

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PREFACE

This book is written for the serious student, probably at the graduate level, who is interested in obtaining an understanding of the theory of Fourier and Laplace transforms, together with the basic theory of functions of a complex variable, without which the transform theory cannot be understood. No prior knowledge other than a good grounding in the calculus is necessary, although undoubtedly the material will have more meaning in the initial stages for the student who has the motivation provided by some understanding of the simpler applications of the Laplace transform. Such prior knowledge will usually be at an introductory level, having to do with the mechanical manipulation of formulas. It is reasonable to begin a subject by the manipulative approach, but to do so should leave the serious student in a state of unrest and perhaps mild confusion. If he is alert, many of the manipulative procedures will not really make sense. If you have experienced this kind of confusion and if it bothers you, you are ready to profit from a study of this book, which occupies a position between the usual engineering treatments and the abstract treatments of the mathematicians. The book is intended to prepare you for creative work, not merely to solve stereotyped problems. The approach is intended for workers in an age of mature technology, in which the scientific method occupies a position of dominance. Because of the heavy emphasis on interpretation and because of the lack of generality in the proofs, this should be regarded as an engineering book, in spite of the extensive use of mathematics.

The highly personal aspect of the learning process makes it impossible for an author to write a book that is ideal for anyone except himself. Recognition of this reality provides the key to how you can benefit most from a book such as this. Probably you will want first to search for the main pattern of ideas, with the details to be filled in at such time as your interest is aroused. Learning is essentially a random process, and an author cannot insist that events in your program of learning will occur in any predictable order. Therefore, it is recommended that you remain alert to points of interest, and particularly to points of confusion. To acknowledge that a concept is not fully understood is to recognize it as a point of interest. It is suggested that you give due respect to such a point, at the time it cries for attention, without regard for whether it is the next topic in the book—searching for related ideas, referring to other texts, and, above all, experimenting with your own ideas. There is such a wealth of interrelatedness of topics that, if you do this with complete intellectual honesty, you will eventually find that you have more than covered the text, without ever having read it in continuous fashion from cover to cover.

The text is roughly in two parts. The first part, on functions of a complex variable, begins at a relatively low level. Experience with graduate students in electrical engineering at Syracuse University, over a period of five years during which the first eleven chapters of this material were used in note form, indicates that the approach is acceptable to most beginning graduate students. The level of difficulty gradually increases throughout the book, and the material beyond Chapter 7 attains a relatively high degree of sophistication. However, it is anticipated that with a gradual increase in your knowledge the material will present an aspect of approximately constant difficulty.

The material on functions of a complex variable is quite similar to many of the standard beginning engineering-oriented texts on the subject, except perhaps for the amount of interpretation and illustrative material. One other difference will be noticed immediately: the use of $s = \sigma + j\omega$ instead of the usual $z = x + iy$. To use j in place of i is established practice in engineering literature and probably is not controversial. The choice of s, σ, and ω in place of z, x, and y was a calculated risk in terms of reader reaction. It provides a unity in this one book, but it will necessitate a symbol translation when comparing with other books on function theory. My apologies are offered to anyone for whom this is a nuisance.

A few suggestions are offered here, to both student and teacher, as to what material might be considered superfluous in an initial course of study. Chapter 1 provides motivation and a perspective viewpoint. It will not serve all students equally well, possibly being too concise for some and too elementary for others. It has no essential position in the stream of continuity and therefore can be omitted. Chapter 2 gives the main introductory concepts and is essential. Chapter 3 should be covered to the extent of firmly establishing the geometrical interpretation of a function of a complex variable as a point transformation between two planes, and the basic ideas of conformality of the transformation as related to analyticity of the function. However, on first reading it may be advisable not to go into details of all the examples given. Much of this is reference material.

Chapters 4 and 5 are very basic and should not be omitted. Chapter 6 bears a relation to the general text material similar to that of Chapter 3.

Some knowledge of multivalued functions is certainly essential, but the student should adjust to his own taste how much detail and how many practical illustrations are appropriate. Much of this chapter may be regarded as reference material. Chapter 7, the last of the chapters devoted to function theory, consists almost completely of reference material pertinent to network theory, and can be omitted without loss of continuity. In fact the encyclopedic nature of this chapter causes some of the topics to appear out of logical order.

Chapter 8 contains background on certain properties of integrals, particularly improper integrals, in anticipation of applications in the later chapters. This chapter deals with difficult mathematical concepts and, compared with the standards of rigor set in the other chapters, is largely intuitive. The main purpose is to alert the reader to the major problems arising when an improper integral is used to represent a function. The chapter can be skipped without loss of continuity, but at least a cursory reading is recommended, followed by deeper consideration of appropriate parts while studying the later chapters.

Chapters 9 and 10 form the core of the second part of the book—the Fourier and Laplace transform theory. In the transition from the Fourier integral to the one-sided Laplace integral, the two-sided Laplace integral is introduced, on the argument that conceptually the two-sided Laplace integral lies midway between the other two. This seems to smooth the way for the student to negotiate the subtle conceptual bridge between the Fourier and Laplace transform theories. If the Laplace transform theory is to be understood at the level intended, there seems to be no alternative but to include the two-sided Laplace transform.

The theory of convolution integrals presented in Chapter 11 is certainly fundamental, although not wholly a part of Laplace transform theory, and therefore should be included in a comprehensive course of study. The remaining chapters deal with special topics, and each has its roots in the all-important Chapter 10. Chapter 12 is essentially a continuation of Chapter 10 and is primarily a reference work. The practical applications treated in Chapter 13 provide a brief summary of the theory of linear systems and preferably should be studied together with, or following, a course in network theory. Otherwise the treatment may be too abstract. However, taken at the proper time, it can be helpful in unifying the ideas about this important field of application.

In regard to Chapter 14, on impulse functions, a critical response from some readers is anticipated by saying that this chapter represents one particular viewpoint. Many will say that it labors the point and that all the useful ideas contained therein can be reduced to one page. This is a matter of opinion, and it is thought that a significant number of students can benefit from this analysis. Everyone knows that impulse functions are not functions in the true sense of the word, and this chapter has something to say about this question, casting the usual results in such a form that no doubts can arise as to the meaning. A knowledge of the customary formalisms associated with impulse functions and their symbolic transforms represents a bare minimum of accomplishment in this chapter.

Finally, Chapters 15 and 16 on periodic functions and the Z transform are related and provide background material for many of the practical applications which you will probably study elsewhere. The justification for including these chapters is to be found in the desire to present this fundamental applied material with the same degree of completeness as the Laplace transform itself.

No particular claim is made for originality in the basic theory, other than in organization and details of presentation. Nor is it claimed that the proofs are always as short or as elegant as possible. The general criterion used was to select proofs that are realistically straightforward, with the hope that this would ensure a high degree of intellectual honesty, while always keeping in touch with simple concepts.

In its preliminary versions, this material has been taught by about twenty different colleagues. All of these persons have made helpful suggestions, and a list of their names would be too long to give in its entirety. However, I would like to single out Professors Norman Balabanian, David Cheng, Harry Gruenberg, Richard McFee, Fazlollah Reza, and Sundaram Seshu as having been especially helpful. Also, Professor Rajendra Nanavati and Messrs. Joseph Cornacchio and Robert Richardson deserve special acknowledgment for reading and constructively criticizing the entire manuscript. Similar acknowledgment is made to Professors Erik Hemmingsen and Jerome Blackman, of the Syracuse Mathematics Department, for their careful reviews of Chapters 8 through 12. Finally, my sincere thanks go to Miss Anne L. Woods for her skill and untiring efforts in typing the various versions of the notes and the final manuscript.

Wilbur R. LePage

CONTENTS

CHAPTER 1

CONCEPTUAL STRUCTURE OF SYSTEM ANALYSIS

1-1. Introduction. It is worthwhile for a serious student of the analytical approach to engineering to recognize that one important facet of his education consists in a transition from preoccupation with techniques of problem solving, with which he is usually initially concerned, to the more sophisticated levels of understanding which make it possible for him to approach a subject more creatively than at the purely manipulative level. Lack of adequate motivation to carry out this transition can be a serious deterrent to learning. This chapter is directed at dealing with this matter. Although it is assumed that you are familiar with the Laplace transform techniques of solving a problem, at least to the extent covered in a typical undergraduate curriculum, it cannot be assumed that you are fully aware of the importance of functions of a complex variable or of the wide applicability of the Laplace transform theory.

Since motivation is the primary purpose of this chapter, for the most part we shall make little effort to attain a precision of logic. Our aim is to form a bridge between your present knowledge, which is assumed to be at the level described above, and the more sophisticated level of the relatively carefully constructed logical developments of the succeeding chapters. In this first chapter we briefly use several concepts which are reintroduced in succeeding chapters. For example, we make free use of complex numbers in Chap. 1, although they are not defined until Chap. 2. Presumably a student with no background in electric-circuit theory or other applications of the algebra of complex numbers could study from this book; but he would probably be well advised to start with Chap. 2.

Most of Chap. 1 is devoted to a review of the roles played by complex numbers, the Fourier series and integral, and the Laplace transform in the analysis of linear systems. However, the theory ultimately to be developed in this book has applicability beyond the purely linear system, particularly through the various convolution theorems of Chap. 11 and the stability considerations in Chaps. 6, 7, and 13.

1-2. Classical Steady-state Response of a Linear System. A brief summary of the essence of the sinusoidal steady-state analysis of the response of a linear system requires a prediction of the relationship between the magnitudes A and B and initial angles α and β for two functions such as

$$\begin{aligned} v_a &= A \cos(\omega t + \alpha) \\ v_b &= B \cos(\omega t + \beta) \end{aligned} \tag{1-1}$$

where v_a, for example, is a driving function* and v_b is a response function. From a steady-state analysis we learn that it is convenient to define two complex quantities

$$V_a = Ae^{j\alpha} \qquad V_b = Be^{j\beta} \tag{1-2}$$

which are related to each other through a system function $H(j\omega)$ by the equation

$$V_b = H(j\omega)V_a \tag{1-3}$$

$H(j\omega)$, a complex function of the real variable ω, provides all the information required to determine the magnitude relationship and the phase difference between input and output sinusoidal functions. Presently we shall point out that $H(j\omega)$ also completely determines the nonperiodic response of the system to a sudden disturbance.

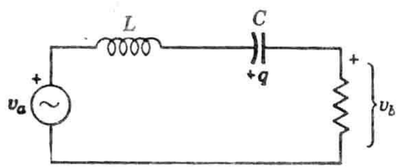

FIG. 1-1. A physical system described by the function in Eq. (1-4).

In the example of Fig. 1-1, the $H(j\omega)$ function is

$$H(j\omega) = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \tag{1-3a}$$

$$H(j\omega) = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}} e^{j[\pi/2 - \tan^{-1} \omega RC/(1-\omega^2 LC)]} \tag{1-3b}$$

Equation (1-3a) emphasizes the fact that $H(j\omega)$ is a rational function (ratio of polynomials) of the variable $j\omega$, and Eq. (1-3b) places in evidence the factors of $H(j\omega)$ which are responsible for changing the magnitude and angle of V_a, to give V_b. Evaluation of the steady-state properties of a system is usually in terms of magnitude and angle functions given in Eq. (1-3b), but the rational form is more convenient for analysis.

This brief summary leaves out the details of the procedure for finding $H(j\omega)$ from the differential equations of a system. It should be recognized that $H(j\omega)$ is a rational function only for systems which are described by ordinary linear differential equations with constant coefficients.

1-3. Characterization of the System Function as a Function of a Complex Variable. The material of the preceding section provides our first

* The terms driving function, forcing function, and excitation function are used interchangeably in this text.

point of motivation for a study of functions of a complex variable. In the first place, purely for convenience of writing, it is simpler to write

$$H(s) = \frac{RCs}{1 + RCs + LCs^2} \tag{1-4}$$

which reduces to Eq. (1-3a) if we make the substitution $s = j\omega$. However, wherever we write an expression like this, with s indicated as the variable, we understand that s is a *complex variable*, not necessarily $j\omega$. In fact, throughout the text we shall use the notation $s = \sigma + j\omega$. Another advantage of Eq. (1-4) is recognized when it appears in the factored form

$$H(s) = \frac{(R/L)s}{(s - s_1)(s - s_2)} \tag{1-5}$$

Carrying these ideas a bit further, we observe that the general steady-state-system response function can be characterized as a rational function

$$H(s) = K\,\frac{(s - s_1)(s - s_3) \cdots (s - s_n)}{(s - s_2)(s - s_4) \cdots (s - s_m)} \tag{1-6}$$

Various systems differ with respect to the degree of numerator and denominator of Eq. (1-6), in the factor K, and in the locations of the critical values s_1, s_2, s_3, etc. The quantities s_1, s_3, etc., in the numerator are called zeros of the function, and the corresponding s_2, s_4, etc., in the denominator are poles of the function. In general, these critical values of s, where $H(s)$ becomes either zero or infinite, are complex numbers, emphasizing the need to deal with complex numbers in the analysis of a linear system.

Equation (1-6) provides an example of the importance of becoming accustomed to thinking in terms of a *function* of a complex variable, since, with $s = j\omega$ and ω variable, this function represents the variation of system response as a function of frequency. In particular, the variation of response magnitude with frequency is often important, as in filter design; and Eq. (1-6) provides a convenient vehicle for obtaining this functional variation. Geometrically, each factor in the numerator or denominator of Eq. (1-6) has a magnitude represented graphically by line AB in Fig. 1-2a, shown for the particular case where s_k is a negative real number. Except for the real multiplying factor K, for any complex value of s the complex number $H(s)$ has a magnitude which can be calculated as a product and quotient of line lengths like AB in the figure and an angle which is made up of sums and differences of angles like α_k. Thus, a plot in the complex s plane provides a pictorial aid in understanding the properties of the function $H(s)$. In particular, steady-state response for variable frequency is characterized by allowing point s to move along the vertical axis.

This formulation is also helpful when we are concerned with variation of the magnitude $|H(j\omega)|$ as a function of ω. The function $H(s)H(-s)$ plays a central role in this question. $H(-s)$ is made up of products and quotients of factors like $-s - s_k$, one of which is portrayed by magnitude

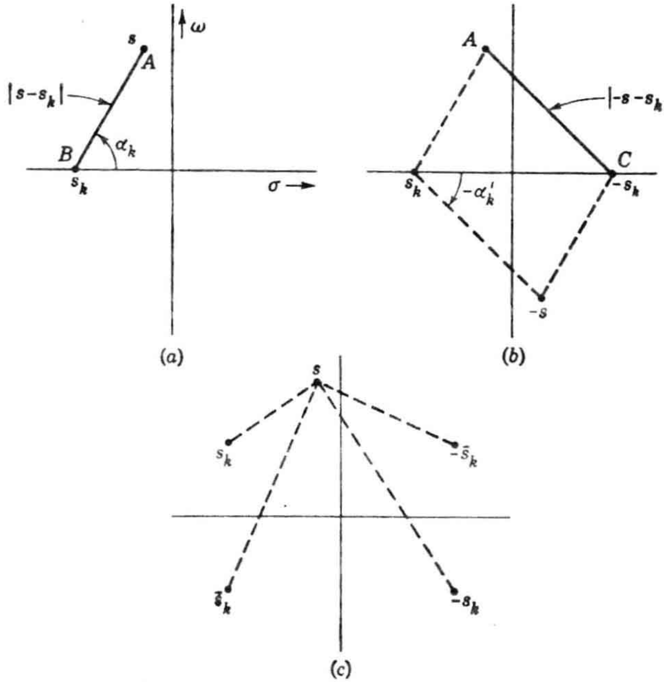

FIG. 1-2. Geometrical interpretation of factors in the numerator and denominator of $H(s)$, $H(-s)$, and $H(s)H(-s)$. (*a*) A factor of $H(s)$; (*b*) a factor of $H(-s)$; (*c*) factors of $H(s)H(-s)$ due to a pair of conjugate zeros or poles of $H(s)$.

$|-s - s_k|$ and angle α'_k in Fig. 1-2*b*. Thus, if each s_k is real, $H(s)H(-s)$ is formed from the product and quotient of factors like

$$|s - s_k|\,|-s - s_k|e^{j(\alpha_k + \alpha_k')}$$

The geometry of Fig. 1-2*a* makes it evident that when $s = j\omega$ (placing s on the vertical axis) the sum of angles $\alpha_k + \alpha'_k$ is zero and therefore $H(j\omega)H(-j\omega)$ is real. Furthermore, $AB = AC$ when $s = j\omega$, and therefore $H(j\omega)H(-j\omega)$ is the square of the magnitude of $H(j\omega)$. It can be shown, from physical considerations, that, if s_k is complex, the factor $s - s_k$ is accompanied by a companion factor $s - \bar{s}_k$, where $\bar{s}_k$ is the complex conjugate of s_k, as illustrated in Fig. 1-2*c*.* In that case, both factors

* A bar above a symbol designates its complex conjugate. Conjugates bear to each other the relation shown in the figure, having the same real components and imaginary components of opposite signs. The conjugate is defined and discussed in Chap. 2.

are considered together, with the conclusion that the product of four factors $(s - s_k)(s - \bar{s}_k)(-s - s_k)(-s - \bar{s}_k)$ is real when $s = j\omega$. Thus, since K is real, we find generally that the function $H(s)H(-s)$ is a function of a *complex variable* which has the peculiar property of *being real* when $s = j\omega$ and furthermore of being the square of the magnitude of $H(j\omega)$. We summarize by writing

$$|H(j\omega)|^2 = H(s)H(-s)\Big|_{s=j\omega} \tag{1-7}$$

Equation (1-4) can be used for an illustration, where

$$H(-s) = \frac{-RCs}{1 - RCs + LCs^2} \tag{1-8}$$

giving

$$H(s)H(-s) = \frac{-R^2C^2s^2}{(1 + LCs^2)^2 - R^2C^2s^2} \tag{1-9}$$

When $s = j\omega$, since $(j\omega)^2 = -\omega^2$, we obtain

$$H(j\omega)H(-j\omega) = \frac{R^2C^2\omega^2}{(1 - LC\omega^2)^2 + R^2C^2\omega^2} \tag{1-10}$$

which is the square of the magnitude factor in Eq. (1-3*b*).

The function $H(s)H(-s)$ is particularly important in the design of filter and corrective networks because of the property just demonstrated. Again we can say that analytical work is easier if we deal with the complex function $H(s)H(-s)$ than if we deal only with the real function $|H(j\omega)|$.

1-4. Fourier Series. The sinusoidal function described in Sec. 1-3 plays a vital role beyond the sinusoidal case for which it is defined. The reason is provided by the Fourier series, whereby a periodic function $v_a(t)$, of angular frequency ω_a, can be described as a sum of sinusoidal components. One way to write the Fourier series for the driving function is

$$v_a(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega_a t + \alpha_n) \tag{1-11}$$

each term of which is like Eqs. (1-1). Assuming that the principle of superposition is applicable, the response is

$$v_b(t) = \sum_{n=0}^{\infty} B_n \cos(n\omega_a t + \beta_n) \tag{1-12}$$

Upon comparing with Eqs. (1-2) and (1-3), it is evident that A_n and B_n and α_n and β_n are related by

$$B_n e^{j\beta_n} = H(jn\omega_a)A_n e^{j\alpha_n} \tag{1-13}$$