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Differential Equations

READING, MASSACHUSETTS · MENLO PARK, CALIFORNIA · LONDON · DON MILLS, ONTARIO

This book is in the
ADDISON-WESLEY SERIES IN MATHEMATICS

LYNN H. LOOMIS, *Consulting Editor*

 Printed in the United States of America. Published simultaneously in Canada. Library of Congress Catalog Card No. 68-15393.

MA 2/73 03951

Elementary Differential Equations

Elementary

ADDISON-WESLEY PUBLISHING COMPANY

preface

In a world that seems to be rapidly filling with mathematics books, the appearance of still another requires at least an explanation if not an apology, especially when the book at issue is on such a venerable and thoroughly exposed subject as elementary differential equations. Of course, the very existence of this book in published form stands as an assertion of the authors' conviction that it does in fact merit publication. But this hardly justifies the deed; hardly answers the just and reasonable question: Why another book on differential equations?

For those familiar with the subject, that is, for those who teach it rather than study it, a glance at the table of contents should provide the answer. We have attempted to write a book which rescues the traditional first course in differential equations from the wasteland of unrelated techniques and dreary formalism in which it has all too long been lost. We are thereby acting upon our prejudice that applications of mathematics cannot be taught in isolation from the mathematical theory that supports them—not, at least, without crippling the student's ability to respond to the changing techniques of his own field and rendering him powerless to communicate with the next generation of mathematicians and scientists who will be introducing new techniques. Our book, in short, attempts to restore to the subject of differential equations a measure of mathematical relevance and, hopefully, elegance too. As such it qualifies as a "modern" introductory text on differential equations.

There are, of course, many ways in which this can be done, all viable, and each possessing its particular merits and shortcomings. The approach which we have chosen uses linear algebra as its starting point, and has the theory of linear differential equations as its major theme. This, we believe, can be defended on numerous grounds. First, the theory of linear differential equations is an especially easy and rewarding application of the ideas discussed in elementary linear algebra. As such it furnishes an excellent example of the way in which apparently unrelated mathematical disciplines reinforce and illuminate one another to their mutual benefit. This is an important lesson for students to learn, and the opportunity to teach it in this context is too valuable to let pass. Second, beginning books on the subject of differential equations inevitably use linear algebra either openly or otherwise. This being the case, there is much to be said for bringing it out into the open where it can be properly treated with tools appropriate to the task. The resulting increase in understanding by the student is certainly worth the added effort. Finally, the student who approaches the subject of differential equations equipped with and willing to use a knowledge of linear algebra is able to proceed much further in a first course than is the student who either lacks this knowledge or else lacks the maturity to aspire to more than a catalog of special techniques. In any event, the latter student will be forced to master this more general approach if he wishes to get beyond the rudiments of the subject, and he is therefore well advised to adopt it from the outset. Such, at least, are the convictions of the present authors, and this book is an attempt to vindicate those convictions in print.

So much for general remarks directed to those who know the subject.

As for those who do not, that is, for students rather than instructors, much of what we have just said will be unintelligible. Unfortunately, this is in the nature of things, and must remain so until the book has been read. As with all authors, we hope that it will be a rewarding and stimulating experience. *Bon voyage!*

Remarks; Useful or Otherwise

I. Since this book assumes nothing more than a knowledge of elementary calculus, but uses linear algebra throughout, the first chapter is devoted to an exposition of the elements of linear algebra. Anyone with a modest knowledge of that subject should be able to begin with Chapter 2, using the first chapter for reference. A similar remark applies to the first half of Chapter 5 where matrices and systems of linear equations are studied. Outside of this, the first seven chapters of the book cover the standard material on ordinary linear differential equations usually discussed in a first course on differential equations.

On the other hand, our use of linear algebra has enabled us to explore a number of topics in the theory of linear differential equations more thoroughly than is customary in a book at this level. For instance, by studying the Wronskian within the context of the theory of linear dependence and independence, we are able to bring the discussion to the point where the Sturm separation and comparison theorems can be easily proved. Similarly, by viewing the method of variation of parameters as a technique for inverting linear differential operators, we are able to introduce the notion of Green's functions and their associated integral operators for initial-value problems. This is particularly rewarding when we come to the study of the Laplace transform since we are then able to present a unified treatment of what all too often strike the student as unrelated techniques for solving differential equations.

In Chapter 8 we take up the study of first-order nonlinear equations, and introduce most of the time-honored methods for solving special classes of equations of this type. Since this material is logically independent of everything that has gone before, the instructor who is possessed of the quaint idea that a course in differential equations ought to begin by studying a differential equation or two can begin with this chapter and then continue with Chapter 1 or 2 as appropriate.

In many respects Chapter 9 is the climax of the book. Here we introduce the notion of fixed points and contraction mappings, and then go on to prove the existence and uniqueness theorems which have been used as the theoretical basis for much of our earlier work. Finally, in Chapter 10 we introduce the notion of stability, and classify the stability (or instability) of the solutions of plane autonomous systems of differential equations.

II. The internal reference system used in the text works as follows: Items in a particular chapter are numbered consecutively as, for example, (3–1) to (3–100). The first numeral refers to the chapter in question, the second to the numbered item within that chapter.

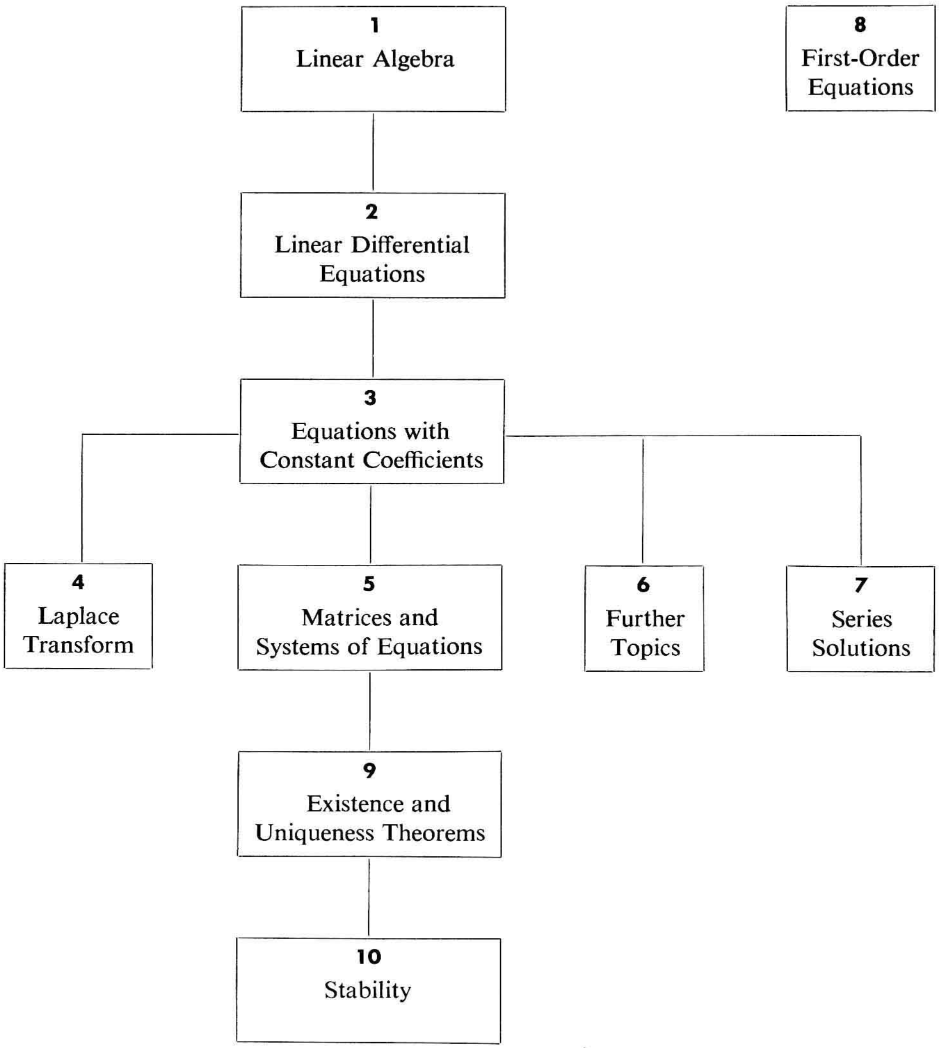
III. Throughout the book we have followed the popular device of indicating the end of a formal proof by the mark ▌ in the belief that students derive a certain comfort from a clearly visible sign telling them how far they must go before they can relax. As usual, sections marked with an asterisk can be omitted without courting disaster, while problems so marked are invitations to just that.

IV. Finally, the authors would like to extend their thanks to Professor Fred W. Perkins who provided the answers to many of the problems, to Dartmouth College's time-shared computer which contributed a number of the figures, and to the Addison-Wesley staff who saw the book through press.

December, 1967

D.L.K.
R.G.K.
D.R.O.

logical interdependence of chapters



contents

Chapter 6 Further Topics in the Theory of Linear Differential Equations*

Chapter 7 Series Solutions of Linear Differential Equations

Chapter 8 First-Order Differential Equations

Chapter 9 Existence and Uniqueness Theorems

1
preliminaries: the elements of linear algebra

1–1 INTRODUCTION

The Cartesian plane of analytic geometry, denoted by $\mathcal{R}^2$, is one of the most familiar examples of what is known in mathematics as a *real vector space*. Each of its points, or *vectors*, is an ordered pair (x_1, x_2) of real numbers whose individual entries, x_1 and x_2, are called the *components* of that vector. Geometrically, the vector $\mathbf{x} = (x_1, x_2)$ may be represented by an arrow drawn from the origin of coordinates to the point (x_1, x_2) as shown in Fig. 1–1.*

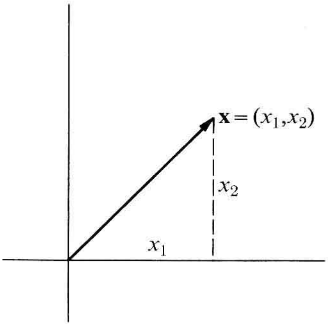


FIGURE 1–1

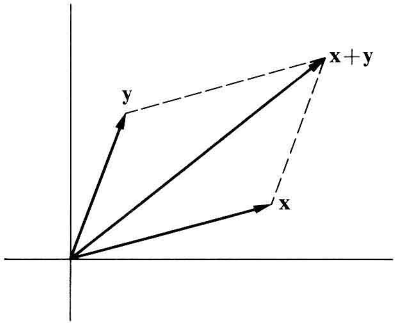


FIGURE 1–2

If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are any two vectors in $\mathcal{R}^2$, then, by definition, their *sum* is the vector

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2) \tag{1–1}$$

obtained by adding the corresponding components of $\mathbf{x}$ and $\mathbf{y}$. The graphical interpretation of this addition is the familiar "parallelogram law," which states that the vector $\mathbf{x} + \mathbf{y}$ is the diagonal of the parallelogram formed from $\mathbf{x}$ and $\mathbf{y}$ (see Fig. 1–2). It follows at once from this definition that vector addition is both

* Throughout this book we shall use boldface type (i.e., **x**, **y**, . . .) to denote vectors.

associative and commutative in the sense that

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}, \tag{1–2}$$

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}. \tag{1–3}$$

Moreover, if $\mathbf{0}$ denotes the vector (0, 0), and $-\mathbf{x}$ the vector $(-x_1, -x_2)$, then

$$\mathbf{x} + \mathbf{0} = \mathbf{x}, \tag{1–4}$$

and

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0} \tag{1–5}$$

for every vector $\mathbf{x} = (x_1, x_2)$. Taken together, Eqs. (1–2) through (1–5) imply that vector addition behaves very much like the ordinary addition of arithmetic.

In addition to being able to add vectors in $\mathcal{R}^2$, we can also form the *product* of a real number α and a vector $\mathbf{x}$. The result, denoted $\alpha\mathbf{x}$, is the *vector* each of whose components is α times the corresponding component of $\mathbf{x}$. Thus, if $\mathbf{x} = (x_1, x_2)$, then

$$\alpha\mathbf{x} = (\alpha x_1, \alpha x_2). \tag{1–6}$$

Geometrically, this vector can be viewed as a "magnification" of $\mathbf{x}$ by the factor α, as illustrated in Fig. 1–3.

The principal algebraic properties of this multiplication are as follows:

$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}, \tag{1–7}$$

$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}, \tag{1–8}$$

$$(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x}), \tag{1–9}$$

$$1\mathbf{x} = \mathbf{x}. \tag{1–10}$$

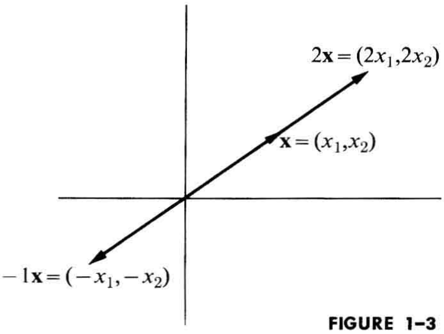


FIGURE 1–3

The validity of each of these equations can be deduced easily from the definition of the operations involved, and save for (1–9), which we prove by way of illustration, they are left for the student to verify. To establish (1–9), let $\mathbf{x} = (x_1, x_2)$ be an arbitrary vector in $\mathcal{R}^2$, and let α and β be real numbers. Then by repeated use of (1–6) we have

$$\begin{aligned}(\alpha\beta)\mathbf{x} &= \big((\alpha\beta)x_1, (\alpha\beta)x_2\big)\\ &= \big(\alpha(\beta x_1), \alpha(\beta x_2)\big)\\ &= \alpha(\beta x_1, \beta x_2)\\ &= \alpha\big(\beta(x_1, x_2)\big)\\ &= \alpha(\beta\mathbf{x}),\end{aligned}$$

which is what we wished to show.

The reason for calling attention to Eqs. (1–7) through (1–10) is that they, together with (1–2) through (1–5), are precisely what make $\mathcal{R}^2$ a real vector space. Indeed, these equations are none other than the axioms in the general definition of such a space, and once this definition has been given, the above discussion constitutes a verification of the fact that $\mathcal{R}^2$ is a real vector space. But before giving this definition, we look at another example.

This time we consider the set $\mathcal{C}[a, b]$ consisting of all real-valued, continuous functions defined on a closed interval $[a, b]$ of the real line.* For reasons which will shortly become clear we shall call any such function a vector, and, following our general convention, write it in boldface type. Thus $\mathbf{f}$ is a vector in $\mathcal{C}[a, b]$ if and only if $\mathbf{f}$ is a real-valued, continuous function on the interval $[a, b]$. Typical examples are such functions as $\sin x$, $\cos x$, and e^x which are vectors in $\mathcal{C}[a, b]$ for any interval $[a, b]$.

At first sight it may seem that $\mathcal{C}[a, b]$ and $\mathcal{R}^2$ have nothing in common but the name "real vector space." However, this is one of those instances in which first impressions are misleading, for as we shall see, these spaces are remarkably similar. This similarity arises from the fact that an addition and multiplication by real numbers can also be defined in $\mathcal{C}[a, b]$ and that these operations enjoy the same properties as the corresponding operations in $\mathcal{R}^2$.

Turning first to addition, let $\mathbf{f}$ and $\mathbf{g}$ be any two vectors in $\mathcal{C}[a, b]$. Then their sum, $\mathbf{f} + \mathbf{g}$, is defined to be the function (i.e., vector) whose value at each point x in $[a, b]$ is the sum of the values of $\mathbf{f}$ and $\mathbf{g}$ at x. In other words,

$$(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x) \qquad (1\text{–}11)$$

(see Fig. 1–4). For example, if $\mathbf{f}$ and $\mathbf{g}$ were the functions $\sin x$ and $\cos x$, their sum $\mathbf{f} + \mathbf{g}$ would be the function $\sin x + \cos x$. In particular the reader should note that $\mathbf{f} + \mathbf{g}$ always belongs to $\mathcal{C}[a, b]$ whenever $\mathbf{f}$ and $\mathbf{g}$ do.

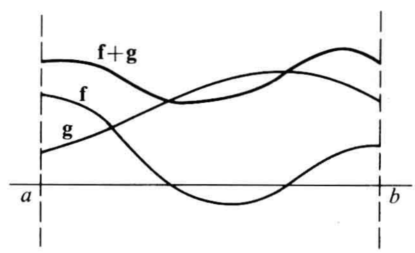


FIGURE 1–4

It is now easy to verify that apart from notation and interpretation Eqs. (1–2) through (1–5) remain valid in $\mathcal{C}[a, b]$. In fact, the equations

$$\mathbf{f} + (\mathbf{g} + \mathbf{h}) = (\mathbf{f} + \mathbf{g}) + \mathbf{h} \qquad (1\text{–}12)$$

and

$$\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f} \qquad (1\text{–}13)$$

follow immediately from (1–11), while if $\mathbf{0}$ denotes the function whose value is

* The *closed interval* $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$; i.e., $[a, b]$ is the interval from a to b, end points included. By contrast, if the end points are not included in the interval, we speak of the *open interval* from a to b, and write (a, b).